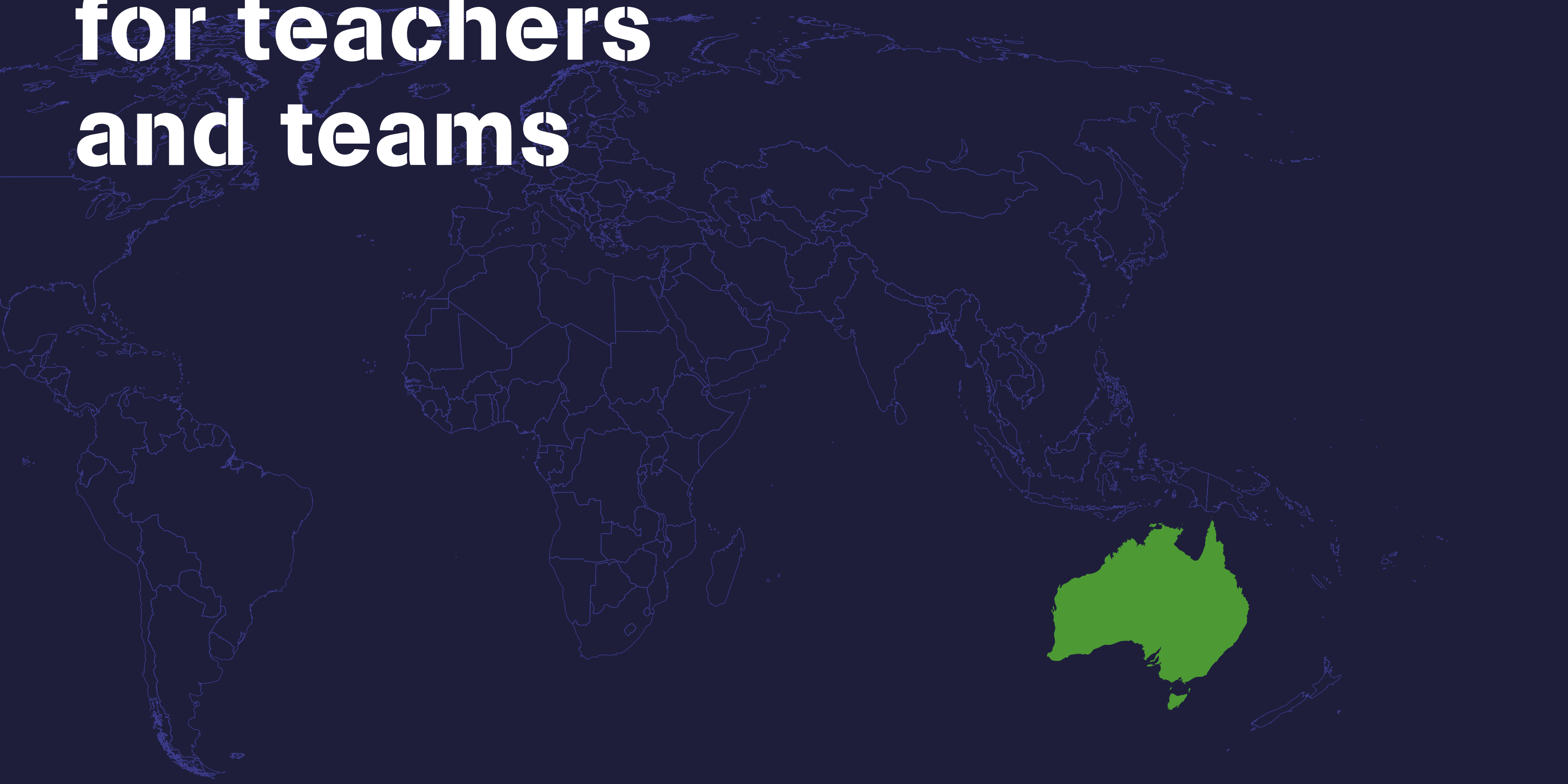
(IM²C international mathematical modeling challenge)

# Mathematical Modelling

## A guidebook for teachers and teams



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ACER

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Design by ACER Creative Services

# Preparing students for mathematical modelling

**Ultimately, only life educates, and the deeper that life, the real world, burrows into the school, the more dynamic and the more robust will be the educational process. That the school has been locked away and walled in as if by a tall fence from life itself has been its greatest failing. Education is just as meaningless outside the real world as is a fire without oxygen, or as is breathing in a vacuum.**

**Vygotsky**[1]

The material here has been developed with two purposes in mind. Firstly, it sets out to provide a basis for the development of workshops geared to those for whom mathematical modelling is a new endeavour. Secondly, it provides support material for teachers already working in the field. The material may be of interest to:

- individuals who want support materials and ideas for teaching modelling up to senior secondary level
- pre-service and in-service education courses featuring components in mathematical modelling
- mentors who intend to enter teams in the IM$^2$C
- teacher organisations, such as the Australian Association of Mathematics Teachers.

These considerations have influenced the way in which the material is structured. The mainstream development works systematically through, and illustrates by example, those features deemed essential for the development of modelling proficiency. This procedure gives autonomy to the user in deciding if, when, and how much of the additional material is appropriate for purposes of the moment. In particular it means that those already working in the field can skip content, or indeed sections, according to their priorities and background. Ultimately it means that users are not forced to engage with everything that is included, while the mainstream development does aim to provide a complete package, beginning from no assumed knowledge and building to sophisticated application of modelling techniques. Finally, problems can often be adapted – simplified or extended according to circumstance, intention, or target audience. At times suggestions along these lines are included – but many others are possible and it is the hope that the contexts provide opportunity for individual initiative and development wherever these are seen as relevant.

This document aims to provide teachers with material that will enable them to prepare students to develop skills of mathematical modelling. It is an espoused goal of curricula that students are enabled to apply the mathematics they learn to address problems in their personal lives, as productive citizens, and in the workplace.

In a specific international initiative to support this ideal, an International Mathematical Modeling[2] Challenge (IM$^2$C) was instigated in 2014. A specific purpose of this material is to provide additional support to assist teachers and schools in choosing and mentoring student teams to participate in the IM$^2$C.

[1] Vygotsky, L.S. (1926/1997). *Educational psychology*. Boca Raton, FL: CRC Press LLC.

[2] Note the American spelling. We use this for the IM²C title as this is used internationally. We have also used American spelling when we are using problems that have been sourced from America.

# What is mathematical modelling?

**It is the mark of an instructed mind to rest satisfied with the precision which the nature of the subject permits and not to seek an exactness where only an approximation to the truth is possible.**

**Aristotle**

Mathematical models are used to help make decisions in a variety of new developments. Governments consult mathematical modellers to predict the consequences of new initiatives, such as a change of the tax system; engineers use mathematical models to build bridges and multi-storey buildings; and social and biological research relies on a great deal of modelling that uses statistics. For a good article on the importance and use of mathematical modelling, see '5 Reasons to Teach Mathematical Modeling', by mathematics professor Rachel Levy.3 This is well worth the read for both teachers and students.

Children are natural modellers. For example they know that often '1 = 2' provides an equitable solution to the problem of sharing lollies of different sizes (one big lolly equals two smaller ones). Yet formal schooling so often robs initiative in the name of conformity. As with all those who value mathematical modelling as real world problem solving, the IM$^2$C program begins from an assumption that mathematics is everywhere in the world around us – a challenge is to identify its presence, access it, and apply it productively. The OECD, the American Common Core Standards Initiative, the Australian Curriculum, and other national curricula all avow that students should have a mathematical preparation which equips them to use their acquired knowledge in their personal lives, as citizens, and in the workplace.

Such a purpose implies two intersecting goals. Firstly, to develop a systematic and successful approach to addressing individual problems located in real-world settings, and secondly, through this means, to enable students cumulatively to become real-world problem solvers. The latter means that they not only can address productively problems set by others, but become able to identify and address problems that matter to them.

In the field of education a variety of meanings have been attached to the term mathematical modelling. At a very simplistic level the term 'mathematical model' has been used in the sense of a formula. Another common notion is associated with mathematical modelling used as a vehicle to achieve other curricular goals. As such the purpose is not the construction of mathematical models to solve problems per se, but rather to contrive the embedding of some pre-determined mathematics in a contextual setting as a mechanism for the learning of certain mathematical concepts, procedures and so on.

Mathematical modelling approached as content entails the construction of mathematical models of natural and social phenomena that are problem driven, and where the choice of relevant mathematics is itself part of the solving process. It is this type of modelling that the IM$^2$C initiative seeks to promote and reinforce.

[3] Levy, R. (2015, May 5). 5 reasons to teach mathematical modelling. *American Scientist*. https://www.americanscientist.org/blog/macroscope/5-reasons-to-teach-mathematical-modeling

**Mathematical modelling is:**

- a process in which real-life situations and relations in these situations are expressed by using mathematics (Haines & Crouch[4]), or
- a cyclical process in which real-life problems are translated into mathematical language, solved within a symbolic system, and the solutions tested back within the real-life system (Verschaffel, Greer, & De Corte[5]).

In both instances, mathematical models are seen to move beyond the physical characteristics of a real-life situation to examine its structural features through mathematics; it entails the construction of mathematical models of natural and social phenomena that are problem-driven, and where the choice of relevant mathematics is itself part of the solving process.

For an introductory example of the process for construction of a mathematical model, see 'Adapting a recipe' on p. 9. This includes a spatial representation mapping the modelling process for a simple problem.

To help ensure that we are all on the same theoretical page, we refer to the approach to modelling summarised in a study by the International Commission on Mathematical Instruction (see Figure 1).[6]

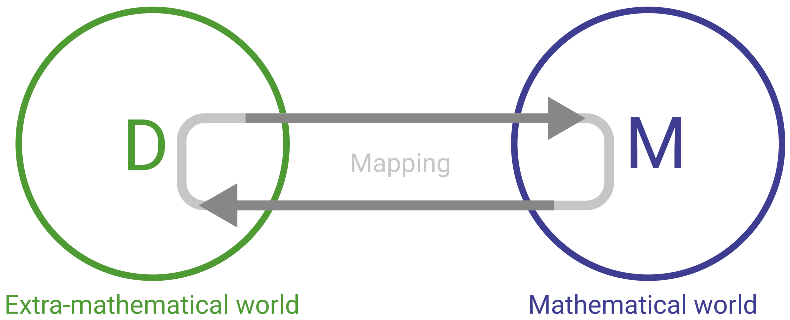
**Here a mathematical model is represented in three parts:**

1. an extra-mathematical domain (D)
2. some mathematical domain (M)
3. mapping from the extra-mathematical domain (D) to the mathematical domain (M), resulting in outcomes that are translated back to the extra-mathematical domain (D).

The extra-mathematical world is a useful way of representing that subset of the wider 'real world' that is relevant to a particular problem. By real world we mean everything that is to do with nature, society, or culture, including everyday life, as well as school or university subjects, and scientific or other scholarly disciplines different from mathematics.

Objects, relations, phenomena, assumptions, questions, etc. in D are identified and selected as relevant for the purpose and situation and are then mapped – translated – into objects, relations, phenomena, assumptions, questions, etc. pertaining to M. Within M, mathematical deliberations, manipulations and inferences are made, the outcomes of which are then translated back to D and interpreted as conclusions concerning that domain. Within the above activity, the use of technology is relevant whenever it can enhance the mathematical process – which is often. This so-called modelling cycle, (to be elaborated and illustrated shortly) may be iterated several times, on the basis of validation and evaluation of the model in relation to the domain, until the resulting conclusions concerning D, are satisfactory in relation to the purpose of the model construction.

*Figure 1: The iterative mathematical modelling cycle*




[4] Haines, C.R., & Crouch, R.M. (2007). Mathematical modelling and applications: Ability and competence frameworks. In W. Blum, P. Galbraith, M. Niss, & H-W. Henn (Eds.), Modelling and applications in mathematics education: The 14th ICMI study (pp. 417–424). New York: Springer.

[5] Verschaffel, L., Greer, B. & De Corte, E. (2002). Everyday knowledge and mathematical modeling of school word problems. In K. Gravemeijer, R., Lehrer, B., Oers, B., van and L. Verschaffel (Eds.), Symbolizing, modeling and tool use in mathematics education (pp. 257–276). Dordrecht, Netherlands: Springer.

[6] Niss, M., Blum, W., & Galbraith, P. (2007). Introduction. In W. Blum, P. Galbraith, M. Niss, & H-W. Henn (Eds.), Modelling and applications in mathematics education: The 14th ICMI study (pp. 3–32). New York: Springer.

# Mathematical modelling framework

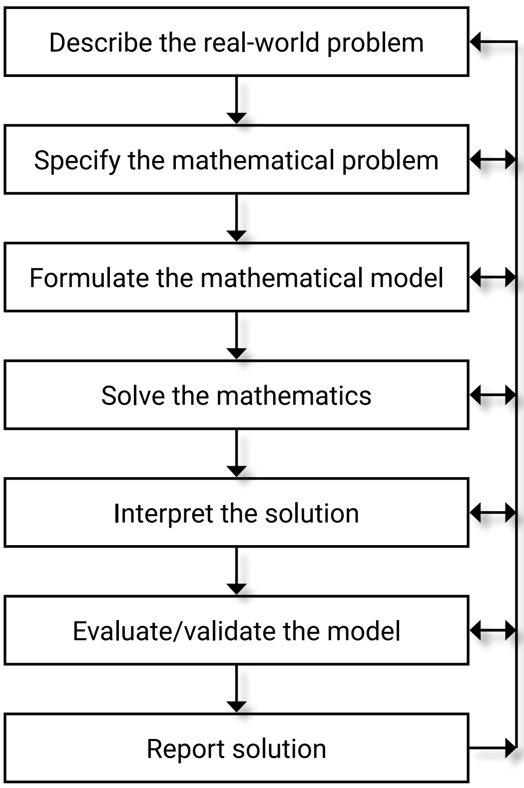
The IM$^2$C operates on the assumption that mathematics is everywhere in the world around us; the challenge is to identify its presence, access it, and apply it productively. The IM$^2$C exists to help students:

- develop a systematic and successful approach to addressing individual problems located in real-world settings, and
- through this development, enable students cumulatively to become effective solvers of real-world problems.

The desire is to produce students who can not only productively address problems set by others, but become able to identify and address problems themselves.

In order to be useful and applicable in practice (both in the context of the IM$^2$C, and more broadly), the cyclic process of modelling referenced above needs to be elaborated in a form that can guide (scaffold) a systematic approach to individual problems. The approach needs to be authentic – that is to be consistent with the way professional modellers approach problems in their field.

*Figure 2: Mathematical modelling framework*




The following list indicates stages in such a process that are sequential in terms of the progress of a solution.

1. **Describe** the real-world problem. Identify and understand the practical aspects of the situation.
2. **Specify** the mathematical problem. Frame the real-world scenario as an appropriate, related mathematical question(s).
3. **Formulate** the mathematical model. Make simplifying assumptions, choose variables, estimate magnitudes of inputs, justify decisions made.
4. **Solve** the mathematics.
5. **Interpret** the solution. Consider mathematical results in terms of their real-world meanings.
6. **Evaluate** the model. Make a judgment as to the adequacy of the solution to the original question(s). Modify the model as necessary and repeat the cycle until an adequate solution has been found.
7. **Report** on success or document how further research could make adjustments and try for a better solution. Communicate clearly and fully your suggestions to address the real-world problem.

The interpretation and evaluation stages indicate the cyclic nature of mathematical modelling.

If the proposed first solution is not an adequate solution to the original question, the problem needs to be readdressed by repeating of earlier stages (stages 3 to 6) in sequence, and this may need to be carried out several times before an adequate solution is found.

Sometimes an extension or refinement of the original problem is suggested by the outcome of a first modelling endeavour. In this instance the question is re-specified, and further cycles of activity are conducted with the new question.

Of particular importance is the realisation that it is the problem and its solution that drive the modelling processes and the choices that the modellers consequently make. For the modeller the activity is usually anything but smoothly cyclic, because checking, testing and evaluating mean that there is frequent movement within and between intermediate stages of the total process – potentially making the development of some models a very challenging exercise. Various versions of the modelling cycle exist, but they all contain the same essentials and ordering of stages.

Figure 2 shows a mathematical modelling framework. The central vertical arrows portray a logical imperative in the progression of a solution through a sequence of stages (e.g., a model cannot be solved before it is formulated), while the horizontal arrows on the right indicate that iterative backtracking may occur between any phases of the modelling cycle, as and when a need is identified.

There are two significant families of modelling: descriptive and prescriptive.

## Descriptive modelling

Descriptive modelling is the most common form of modelling. Usually, a real-world problem is specified, a model formulated and the resulting mathematical results interpreted and evaluated in terms of the needs of the original problem. Where necessary, further (cyclic) modelling activity is conducted until a satisfactory solution is achieved. Descriptive modelling does not necessarily involve validation of the initial solution. For example, modelling phenomena such as global warming or population projections are undertaken without access to future values of variables for purposes of evaluation. Other methods of evaluation are required, for example, completeness of model structure in terms of known influences, accuracy of data values used, and the reasonableness of the resulting outcome. Refinements to and projections from such models will be ongoing, as fresh data become available to improve the quality of inputs and to act as evaluation criteria for earlier models. For example, estimating the impact of variations in fertility and immigration rates, from those assumed in population projection models, are important in testing the robustness of predictions, and hence to prepare for possible future demands on educational, health and social services. Cyclic activity associated with such modelling contexts is continuous and can extend over years. (Within this Guidebook, examples of descriptive modelling problems include Kochel Numbers, Supersize Me, and Population Projections.)

## Prescriptive modelling

In prescriptive modelling, the purpose is not to explain or make predictions about real-world phenomena but to organise or structure a situation, for example, the location of a new power plant. Every prescriptive modelling project also has an inbuilt cyclic expectation; any 'solution' needs to be subjected to sensitivity testing, a procedure that tests how vulnerable the solution is to changes in input variables. At a minimum, this involves recalculations of the model using modified assumptions, slightly amended parameter values, interpretation, and an evaluative judgement of the outcome. If small changes in parameter values create big changes in outcome, the model needs to be totally revisited. (Within this Guidebook, prescriptive modelling problems include Howzat and the 2017 IM2C problem).

The methodology shown in the example problems is one that can be applied to all problems, in that it follows the logic of the structure of Figure 2. It does not mean that solutions using the approach will be identical. Far from it, as different choices for assumptions, parameter values, or mathematical development will lead to their own outcomes. What is common, is that these outcomes must be held accountable in terms of mathematical and modelling criteria.

Following this theme, the solutions to example problems are developed in terms of the structure embedded in Figure 2. This means that specific lesson plans are not produced as there are typically several ways in which assistance and scaffolding can be provided within different phases of the modelling process. In consequence, teachers have autonomy of choice, and the opportunity to support students in ways deemed most suitable to the needs of the moment. Illustrations are provided through the solution processes developed and recorded within the respective problems.

To help learners conceptualise and develop a greater understanding of mathematical modelling, example problems are available for download and distribution.

Throughout these problems, example data sets and information sources are provided. This has been necessary to illustrate workings and development of models. In general, in real-world modelling scenarios, finding appropriate data sources are part of the task. When using these example problems, teachers or mentors may choose to withhold the example data sets provided, and instead direct student activity to make obtaining data part of the modelling task.

The junior modelling problems involve the sorts of mathematical processes that we often undertake automatically. In doing so, we can overlook that we have made assumptions that are so 'obvious' that we did not realise that we had made them. Students should be encouraged to articulate the assumptions and processes required. Those already familiar with basic modelling approaches might wish to skip to later material.

Most students find it hard to write reports, probably because they are given very few opportunities to write any mathematics at all. As the result of many factors, answers seem to be valued more highly than the mathematical arguments leading to them. To help students develop report writing skills, we suggest that they be encouraged to add words to answers. This can start with just the odd word or two of explanation such as 'since … then', 'because of Pythagoras' Theorem …' or 'so the number of apples is 11'.

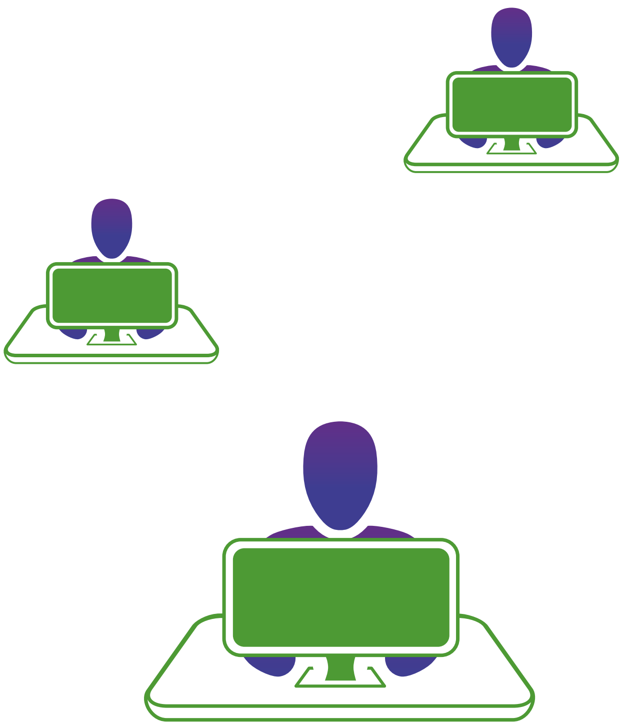
As far as mathematical modelling is concerned, report writing skills are essential and will be of value to students to learn. In many jobs these days, when people often work in multi-skilled teams, it's often necessary for a member of the team to pass information to the others both in oral and written form.

To ease students into report writing, we suggest that students new to modelling are first given practice in writing summaries of no more than a page to each of the modelling activities they do from the following section. This can gradually be extended to full reports as students go through the three stages of examples that follow. To write a full report takes time. We suggest that you don't insist on every report being written in full; students may gain much from working up a bullet summary that could be used to form the notes for an oral report.

Two approaches to the introductory problem 'Adapting a recipe' are provided.

**Approach I** Used as a quick introduction to the modelling process for those with no previous experience of modelling. For this purpose, it is first presented as a pre-modelling exercise, and the modelling process is introduced to formalise what has been done, but often overlooked. The identified process is then applied to other problems. In this approach, the modelling process is highlighted and reporting is restricted to a simple summary of the new recipe. With this approach it is usually possible to move quickly to other problems, often within half an hour.

**Approach II** The second (fuller) approach is used when the complete modelling process is followed by a detailed examination of what is involved in report writing. The example is again used to introduce the modelling process, and illustrate the respective stages. We first show the mathematics used with amplifying words, give a summary, and then finally an idea of what a full report might look like. This approach is geared particularly to introduce and illustrate reporting as a part of the totality.



*If this problem is used as an introduction to modelling (Approach I), give it to the group without further advice. Ask them to work on it for a few minutes and come up with an amended recipe. They will have no knowledge of a modelling process or cycle; however, they will have made assumptions and come up with mathematical answers, which need to be interpreted in practical terms if the recipe is to be realistic. Discussing this will show the need for systematising the process (distribute the modelling diagram at this point). The boxes can be filled out as the group translates and organises what individuals have done. Approach I is useful when the attention is to move quickly to other problems – the main focus is the editing process. Approach II is useful when the construction of a report is a focus, as it illustrated aspects of report writing using a simple problem.*

# Adapting a recipe

## Describe the real-world problem

### Chocolate mousse

185g cooking chocolate

$\frac{1}{4}$ cup hot water

1 teaspoon vanilla essence

5 large eggs

$1\frac{1}{2}$ cups cream

Melt chocolate. Beat eggs.
Whip cream. Fold all ingredients together. Chill in fridge for 1 hour.

Serves 4–6.

## Specify the mathematical problem

Freda Nurke is planning a new cookery book where all her recipes are based around larger/extended families of 6 to 9 people. She would like advice on how she should change the chocolate mousse recipe from her previous book, as shown above.

## Pre-modelling exercise (Approach I or Approach II)

This simple problem requires a recipe to be adapted. Most people would do this on the back of an envelope or even mentally with a calculator to help. We would probably overlook the fact that we had made assumptions and used a process that was so 'obvious' that we did not realise that we had done it.

## Formulate the mathematical model

Group members should be encouraged to recognise and articulate the assumptions they made in their earlier approach to solving the real-world problem.

Introduce the mathematical modelling framework (see Figure 3) and encourage students to map out their response to the problem using this template.

Using the template means that key elements can be represented. This can be useful for structuring a report. In the formulation box, we note essential assumptions that underpin the model development. Identifying them explicitly helps to emphasise their centrality to every modelling enterprise. Similarly, in the interpretation box, real-world practicalities moderate precise mathematical results, a step that we can easily subsume without recognising it.

The three arrows out of box 6 reflect that (in the general case) evaluation may be followed by a report, but may require instead a revisiting of the problem context, the mathematical problem identified – and indeed, other stages of the modelling process.

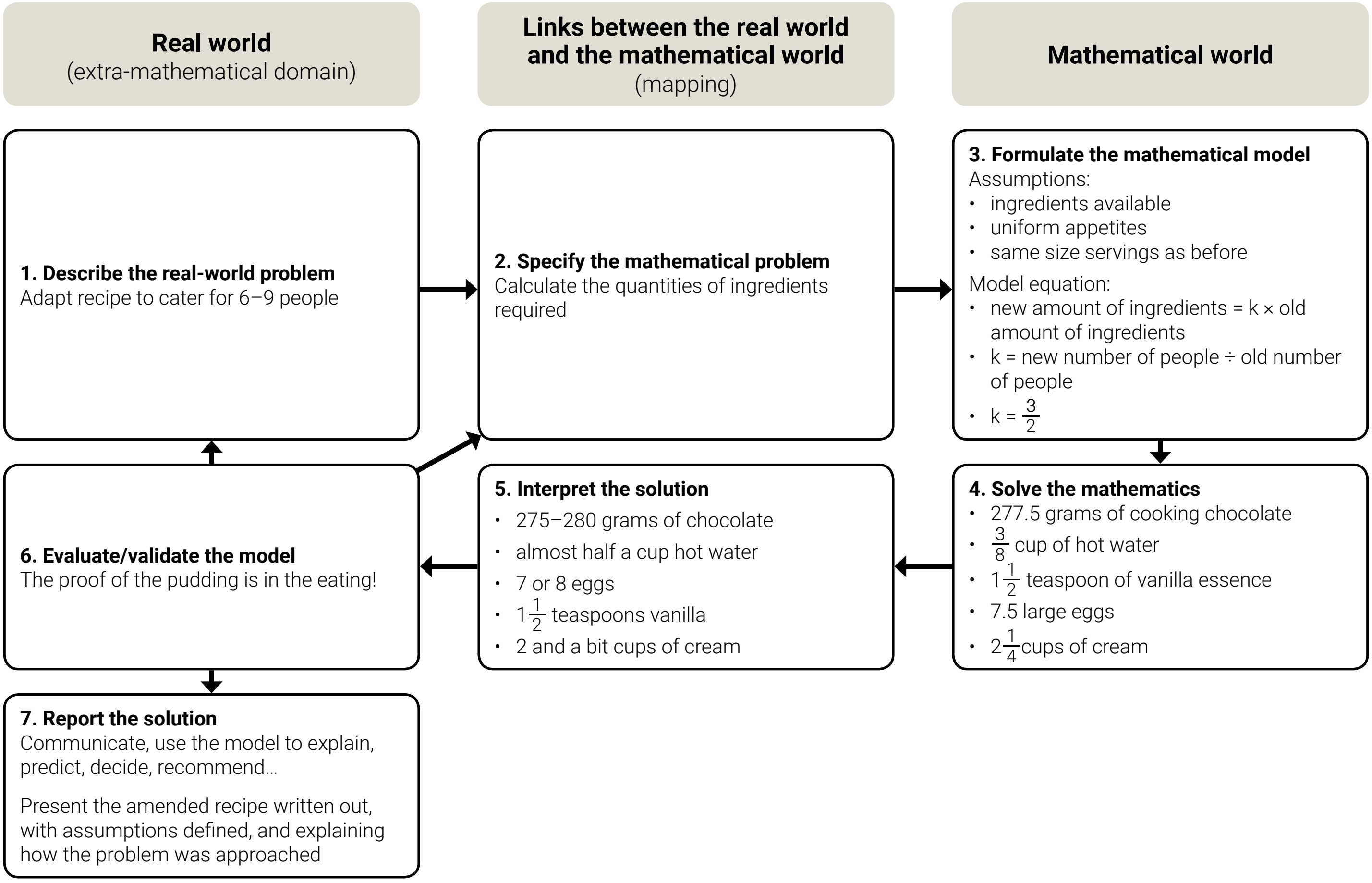
### Situational assumptions

Facilitate students' discussion of what assumptions they will need to make before creating a model. These might include:

- availability of all ingredients
- all the people being served have uniform appetites
- serving size will be the same for the adapted recipe as it was for the original recipe.

Emphasise that the framework is cyclical, not strictly linear. Real-world modelling will often require that students return to earlier steps in the framework to consider new variables, source new data, reassess assumptions and test solutions.

## Figure 3: Adapting a recipe using the mathematical modellling framework




### Mathematical formulation

k = new amount of people ÷ old amount of people

new amount of ingredients = k × old amount of ingredients

$k = 6 \div 4$ or $k = 9 \div 6$

$k = \frac{3}{2}$ or $k = 1.5$

## Solve the mathematics

277.5 grams of cooking chocolate

$\frac{3}{8}$ cup of hot water

$1\frac{1}{2}$ teaspoon of vanilla essence

7.5 large eggs

$2\frac{1}{4}$ cups of cream

## Interpret the solution

Consider the solution in real-world terms. How would you cook with half an egg? Rewrite the ingredients list, for example:

275 or 280 grams of cooking chocolate

almost half a cup of hot water

$1\frac{1}{2}$ teaspoons of vanilla essence

7 or 8 eggs

2 and a bit cups of cream

## Evaluate the model

The exercise can be extended to apply the model to related problems.

What if there are two people coming to dinner? A dozen?

Or, working backwards: If I have half a carton of eggs with which to make mousse, how many people can I invite for dessert?

Will the larger volume of mousse in the amended recipe take longer to chill? If I have less time to prepare, how might I reduce chill time? (For example, use several smaller serving dishes instead of one large dish.)

## Report the solution

Write out a report summary or full report. Examples are shown on the following page.

## Possible extensions

Research real-world shopping limitations. Can I buy exactly 275 grams of chocolate, 7 eggs, 2 cups of cream? How many blocks of chocolate, cartons of eggs, containers of cream would this recipe need? Visit supermarket websites to research product sizes, and write out a shopping list.

Follow up this piece of modelling by a statistical exercise. Make two different lots of mousse and then have a blind testing to see which version people prefer. This could be complicated by using a different recipe for mousse from a different cookbook.

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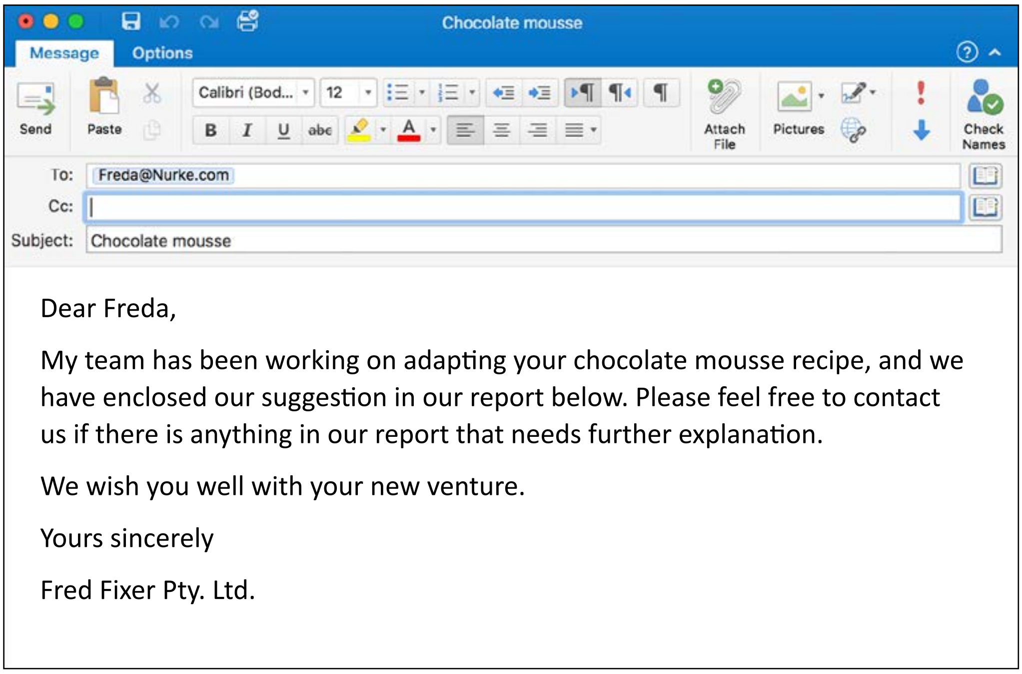

Chocolate mousse

To: Freda@Nurke.com

Cc:

Subject: Chocolate mousse

Dear Freda,

My team has been working on adapting your chocolate mousse recipe, and we have enclosed our suggestion in our report below. Please feel free to contact us if there is anything in our report that needs further explanation.

We wish you well with your new venture.

Yours sincerely

Fred Fixer Pty. Ltd.

### Summary

Using the recipe that you sent us that serves 4 to 6 people, we have produced a mathematical model in which our main assumption is that the new amount of each ingredient will be equal to the old amount of that ingredient, multiplied by the ratio of the new number of people to the number of people in the original recipe. As a consequence of this, we have made a suggestion for the new recipe ingredients to create a dessert to serve 6 to 9 people. This suggestion can be found at the end of our full report.

It is not clear to us, though, how long the new mixture needs to be chilled before it will set satisfactorily. It would be worth your while experimenting in your kitchen to determine an optimal chilling time for your recipe.

In the full report we detail our approach to the problem and discuss other matters that might be considered.

### Full report

Situational assumptions: all ingredients are available; all the people being served have uniform appetites; serving size will be the same for the adapted recipe as it was for the original recipe.

Mathematical assumption: new amount of each ingredient (now denoted k) = old amount of ingredient × new number of people ÷ recipe number of people.

Mathematical model: k = new amount of people ÷ old amount of people

k = 6 ÷ 4 or k = 9 ÷ 6 therefore k = 3/2 or k = 1.5

There are elements in the recipe that may introduce inaccuracies in the adaptation for 6 to 9 people.

For example, there is no international standard for 'cup'. Around the world the volume of a cup varies from 200ml up to nearly 290 ml (see https://en.wikipedia.org/wiki/Cup_(unit)). Further, it is subjective as to what a 'large egg' actually is. If the chef has only small eggs, will 6 be sufficient in the original recipe given? The amount of cooking chocolate to use in the adapted recipe may be unrealistically precise for most kitchen cooks; however, the resulting mousse may taste just as nice or even sweeter if more, rather than less, chocolate is used. And it may be that 1 teaspoon of vanilla essence may do just as well for 4 to 6 people as 6 to 9.

Importantly, it is not clear that the simple ratio method for calculating ingredients is suitable for cooking, and you may want to look further into this. It might be better to make enough for 12 to 18 people and halve this when chilling is complete. The remainder could then be put into the freezer for another day.

Finally, we recommend that you use the amounts proposed below.

Chocolate mousse recipe to serve 6–9 people:

275–280 grams of cooking chocolate

almost half a cup of hot water

$1\frac{1}{2}$ teaspoons of vanilla essence

7 or 8 eggs

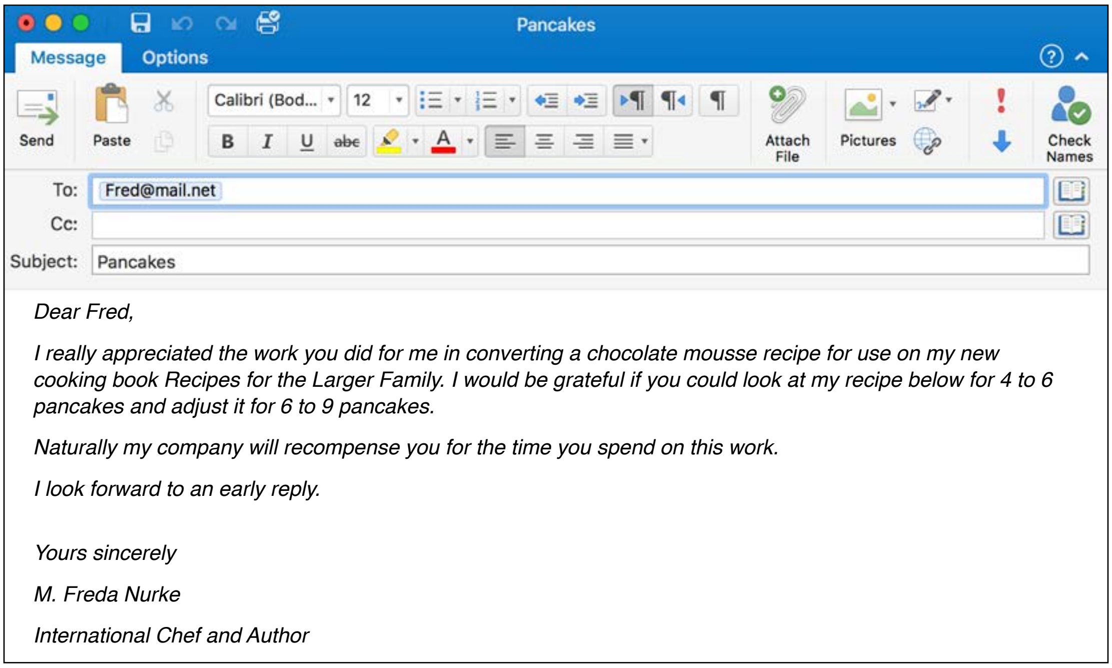
2 and a bit cups of cream

# Junior modelling problems for upper primary, junior or middle secondary

(IM²C international mathematical modeling challenge)

# Pancakes

## Describe the real-world problem




To: Fred@mail.net
Cc:
Subject: Pancakes

*Dear Fred,*

*I really appreciated the work you did for me in converting a chocolate mousse recipe for use on my new cooking book Recipes for the Larger Family. I would be grateful if you could look at my recipe below for 4 to 6 pancakes and adjust it for 6 to 9 pancakes.*

*Naturally my company will recompense you for the time you spend on this work.*

*I look forward to an early reply.*

*Yours sincerely*

*M. Freda Nurke*

*International Chef and Author*

### *Recipe for 4 to 6 pancakes*

1 egg

1 cup self-raising flour

200 ml milk

Mix all ingredients thoroughly so that there are no lumps.

Heat pan until a drop of water dances on the pan.

Add oil to pan.

Add pancake mixture to pan.

Cook pancake on one side till brown, then turn over and cook the other side.

Remove pancake from pan and put on a plate covered by paper towel.

Put into warm oven and add subsequent pancakes to the plate.

## Model the mathematical problem

The modelling activity here is generated by a letter from a satisfied customer, Freda Nurke.

The modelling goal is to produce a recipe for 6 to 9 pancakes from the original recipe for 4 to 6 pancakes. It might be good if the new menu could be tested against the old one for taste.

This example problem is presented without further comment, but can be solved in a similar way to the 'Adapting a recipe' example problem, discussed on the previous pages.

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# Hyperthermia

## Describe the real-world problem

### Danger for children left in hot cars

Monday, December 5, 2015. A King County man will face police questioning over suspicion of negligence after allegedly leaving his two-year-old grandson alone in a car in a shopping centre parking lot on Saturday. Police were forced to smash a window of the locked car to rescue the toddler, who was on the verge of dehydration.

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Hyperthermia, or heat-related illness, occurs when a person's body absorbs more heat from the environment than the body can dissipate through cooling. The human body's cooling mechanisms include perspiration, which is a loss of fluid. Losing a lot of fluid results in dehydration. More than 400 children have died in cars from hyperthermia in the United States since 2005. The children have ranged in age from 5 days to 14 years; more than half have been less than 24 months old.

Source: Null, J. (2016). Heatstroke deaths of children in vehicles. Department of Meteorology and Climate Science, San Jose State University. http://noheatstroke.org

| Child deaths from vehicular heatstroke, United States, by year | | Circumstances leading to child vehicular death |
|---|---|---|
| 2005 | 47 | Child forgotten by adult 54% of cases |
| 2006 | 29 | |
| 2007 | 36 | |
| 2008 | 43 | Child playing unattended in vehicle 29% of cases |
| 2009 | 33 | |
| 2010 | 49 | |
| 2011 | 33 | |
| 2012 | 34 | Child intentionally left in vehicle by adult 17% of cases |
| 2013 | 44 | |
| 2014 | 31 | |
| 2015 | 24 | |

## Specify the mathematical problem

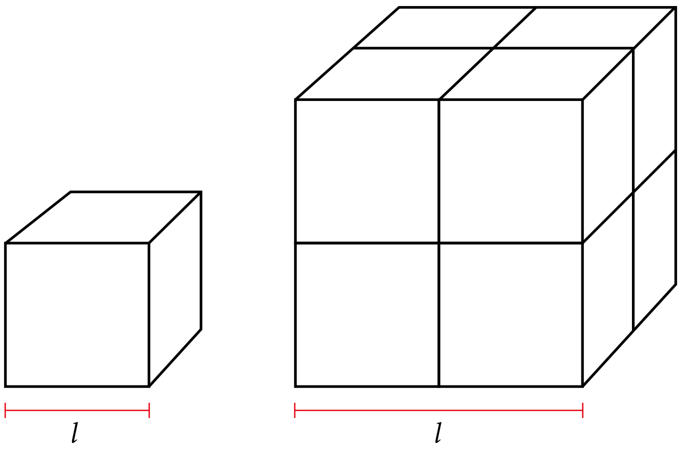
Investigate why small children and animals are so much at risk in locked cars in hot weather.

## Formulate the mathematical model

The rate of fluid loss from a body depends on (is proportional to) its surface area, SA.

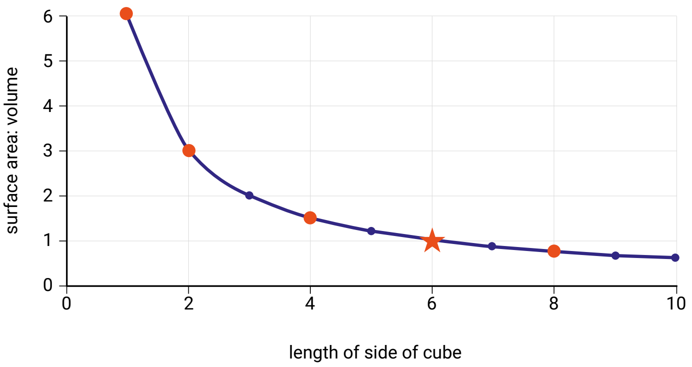
The amount of fluid in a body depends on (is proportional to) its volume, Vol.

So a critical factor is the surface area/volume ratio of the body. We begin by considering blocks of different dimensions.




| | | |
|---|---|---|
| SA small cube = 6 | Vol small cube = 1 | SA/Vol small cube = 6 |
| SA large cube = 24 | Vol large cube = 8 | SA/Vol large cube = 3 |

## Solve the mathematics



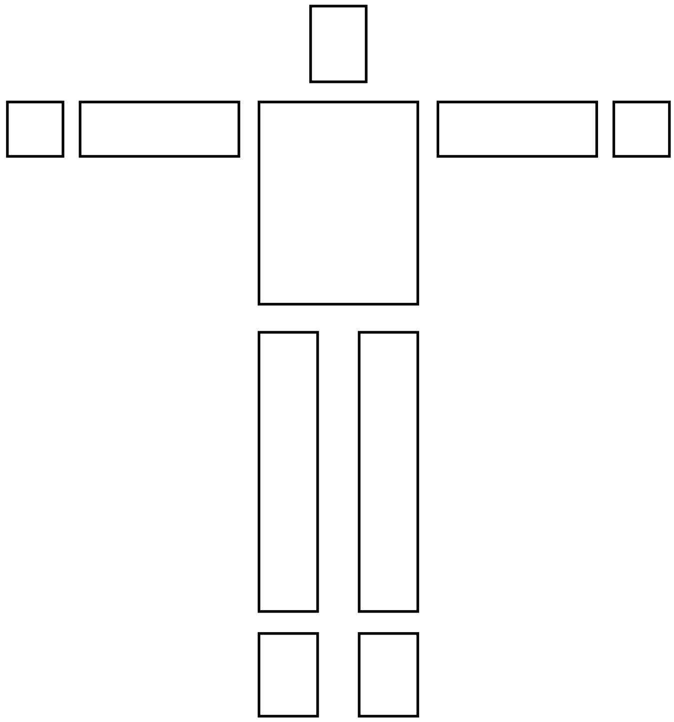

Create a graph plotting the ratio for blocks with lengths from 1 to 10. This shows that smaller cubes have higher surface area/volume ratios than larger cubes. That is, smaller cubes have a greater surface area through which to lose fluid relative to the volume of fluid they have to lose. Smaller cubes will lose fluid at a greater rate than larger cubes.

## Interpret the solution

Applying this logic to the problem of children in cars suggests that children (smaller people) will have a higher surface area/volume ratio than adults (larger people) do. Therefore, they will lose fluid more quickly. Therefore, they are at greater risk of dehydration.

## Evaluate the model

The diagram below suggests how to extend the simple cube approach to construct more elaborate representations, using cuboids, cylinders or spheres, to construct physical models of children, adults, animals and so on. Associated mathematical development will call on knowledge of mensuration. There is opportunity for students to construct representations of animals of choice.



## Report the solution

Students should write out a report following the modelling framework structure.

www.immchallenge.org.au

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# Köchel numbers

Photo credit: Portrait of Mozart by Barbara Krafft

## Describe the real-world problem

> ‘It is sobering to realise that when Mozart was my age he had already been dead for three years.’
>
> — Tom Lehrer, mathematician/comedian

Wolfgang Amadeus Mozart, one of the most influential composers of the Classical era, was born January 27, 1756, in Salzburg, and died December 5, 1793, in Vienna. He composed more than 600 works over his career.

Ludwig von Köchel, Viennese botanist, mineralogist and educator, published an inclusive, chronological catalogue of Mozart’s work in 1862.

Köchel (K) numbers are assigned sequentially according to the date of composition. For example, Mozart’s opera *The Magic Flute* is given the Köchel number 620, and is (approximately) the 620th piece of music Mozart composed.

Compositions completed at the same time are listed K69, K69a, and so on.

## Specify the mathematical problem

A new composition by Mozart completed in April 1784 has come to light. What Köchel number should it be given?

## Formulate the mathematical model

**Data**

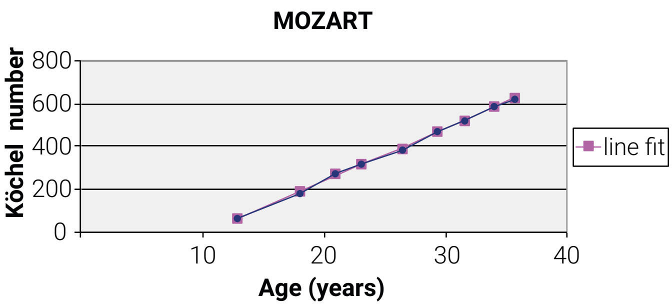
Given the information in the table, find a way of linking Köchel numbers with the dates of completion.

| K | Date completed | Composition |
|---|---|---|
| 65 | Jan 1769 | Dance music |
| 176 | Dec 1773 | Dance music |
| 271 | Jan 1777 | Piano concerto |
| 317 | Mar 1779 | Mass |
| 385 | July 1782 | Symphony: *Haffner* |
| 470 | Apr 1785 | Andante for strings |
| 525 | Aug 1787 | Serenade: *Eine Kleine Nachtmusik* |
| 588 | Jan 1790 | Opera: *Cosi fan tutte* |
| 620 | Sept 1791 | Opera: *Die Zauberflöte* |

From the table above, we translate the completion dates to Mozart's age in years, knowing his birth date. (K65 corresponds to 13.0 and K620 corresponds to 35.7 etc.) These can be added as another column beside 'date completed' in the table. Plotting the points on a grid with Mozart's age along the horizontal axis (x), and Köchel numbers on the vertical axis (y), they are seen to lie approximately along a straight line.

Placing a ruler (by eye) through the points identifies a line which in our judgment best fits the set of points. The line can then be drawn – a so-called 'line of best fit'. The equation to this line will give a link between age and Köchel number.

There are several ways of finding this equation. For example, if our ruled line passes through the endpoints (13.0, 65) and (35.7, 620) of the graph then its equation will be Y = 24.4x − 252.8 (working to one decimal place).




## Solve the mathematics

The date of the new composition (April 1784) corresponds to x = 28.3

Substituting in y = 24.4x − 252.8 gives y = 436.5.

## Interpret the solution

An appropriate Köchel number would be K = 437 or K437a or K437b.

## Evaluate the model

Referring to the table, a composition with K = 437 (April 1784) should lie towards the upper end of the interval between K = 385 (July 1782) and K = 470 (April 1785). It does, so it is reasonable to infer that the model is suited to its purpose.

On the other hand, the model was derived using only nine points. It would be reasonable to repeat the calculations using research to locate say 20 points to use as a starting basis, and to note and comment on any differences.

**Refinement using technology**

Students familiar with graphical calculator technology will likely identify the opportunity to use the regression facility to obtain the line of best fit by technical means. This is of course a legitimate approach, but one which should not be forced on those who are not familiar with the appropriate technology. (Diversion down unfamiliar technological paths has been shown to impede progress within modelling problems.)

In the present case, the application of the linear regression facility is shown in the straight line drawn on the graph above. Its equation is y = 24.8x − 259.3.

This leads to a value of K = 443 which is within 1.5% of the value calculated using eye and hand. The final choice would involve searching out and examining the Köchel numbers already assigned to compositions around the date in question.

## Report the solution

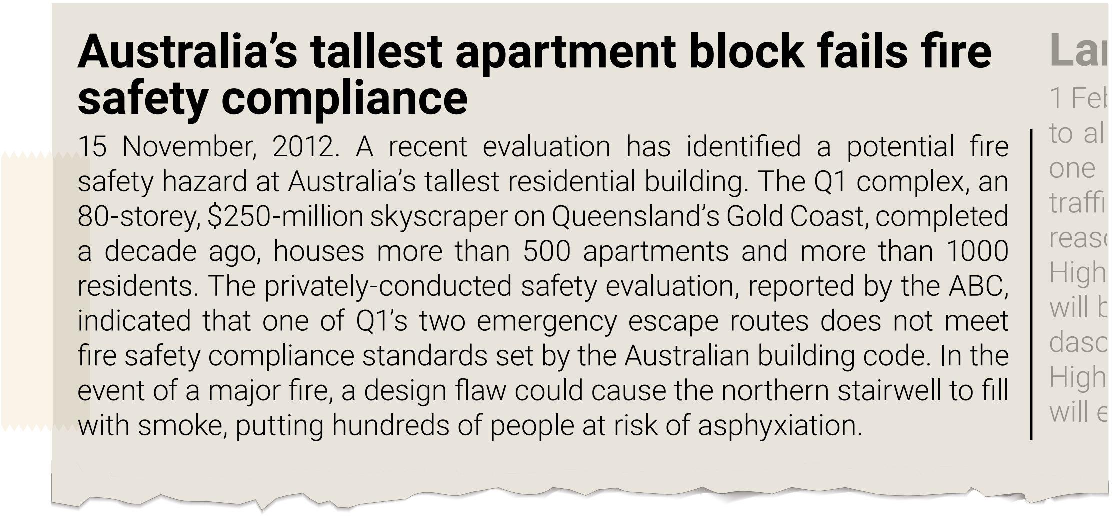
The report should contain the foregoing, perhaps with a summary statement that generalises the usefulness of the model beyond the single case which is featured here. We can note also that a formula such as this can be used generally to check for errors in assigning Köchel numbers, by looking for wayward results when the formula is applied to large numbers of data.

There should also be comment on reservations concerning precision, on account of approximations used in the approach.

# Evacuating

Photo Credit: Jens Sohnrey

## Describe the real-world problem



**Australia's tallest apartment block fails fire safety compliance**

15 November, 2012. A recent evaluation has identified a potential fire safety hazard at Australia's tallest residential building. The Q1 complex, an 80-storey, $250-million skyscraper on Queensland's Gold Coast, completed a decade ago, houses more than 500 apartments and more than 1000 residents. The privately-conducted safety evaluation, reported by the ABC, indicated that one of Q1's two emergency escape routes does not meet fire safety compliance standards set by the Australian building code. In the event of a major fire, a design flaw could cause the northern stairwell to fill with smoke, putting hundreds of people at risk of asphyxiation.

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## Specify the mathematical problem

A bomb threat is received at 2 am and the building must be evacuated. How long would this take if one stairwell cannot be used?

## Formulate the mathematical model

### Data

This is information that defines important aspects of the structure, and can be provided as background or left for students to source (however websites can give conflicting information). This problem is based around estimates rather than exact calculations and the information below is sufficiently accurate to supports this. The Q1 building contains:

- More than 1000 residents (plus staff)
- 76 residential floors
- 527 residential apartments, including a mixture of 1-, 2- and 3-bedroom apartments
- 1331 stairs per stairwell
- 11 lifts
- 2 stairwells

### Situational assumptions

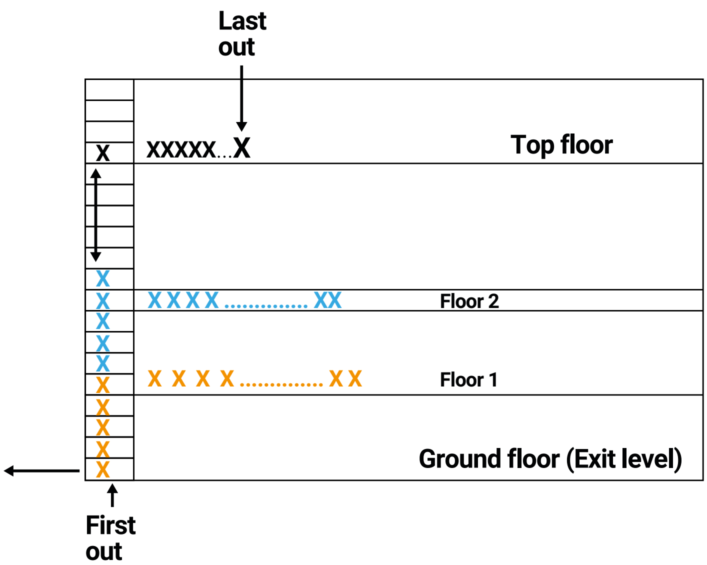
- While building has two exit stairwells, only one is safe.
- Lifts are closed to avoid failure through overcrowding.
- All residents are mobile and hear the evacuation call.
- Building is highly populated and floors have equal numbers of residents.
- 1-, 2- and 3-bedroom apartments are equally distributed.
- Number of stairs between levels is the same throughout.

**Mathematical assumptions**

- Number of floors = 76
- Number of apartments per floor = 527/76 ≈ 7
- Average number of evacuees per apartment (e): e = 2.5* (*Include 0.5 to represent staff presence throughout building)
- Number of evacuees per floor using stairwell (N): N = 7e
- Number of steps per floor/one stairwell (s): s = 665/76 = 8.75 ≈ 9
- Average speed on steps (moving carefully) (v): v = 0.5 steps/sec
- Time delay between successive evacuees when moving (d): d = 1 sec
- Time for first occupant on floor 1 to reach stairs (t): t = 10 sec
- Values of v and d are chosen to reflect a balance between speed and safety.
- Evacuation rate is governed by access to stairs. What happens in corridors is effectively irrelevant because of wait times.

The numbers assigned to the quantities like (e, v, d) need to be estimated, and others (such as 's') inferred. They need to be 'reasonable' in terms of the real context, and can be varied for different calculations.

For example, the average number of evacuees per apartment has been chosen with a fairly 'full' building in mind. Average speed on steps, and time delay between people moving, need to take into account that people crashing into one another must be avoided, while otherwise moving as quickly as possible. Values here can be agreed by discussion, and/or acting out. (Students have used thought experiments effectively here.) Similarly, selection of other numerical values needs to be supported by discussion.




Total time (T) for evacuation = time for first occupant on floor 1 to leave building + delay until last occupant can move + time for last occupant to leave building (this needs thinking about).

Show that:

$T = (t + s/v) + (fN - 1)d + fs/v$ where $N = 7e$

## Solve the mathematics

e = 2.5; f = 76; s = 9; v = 0.5; d = 1; t = 10 gives T ≈ 45.4 min

For a lightly populated building assume e = 1.5; for faster or slower steps vary 'v'; for different separation distances between moving evacuees vary 'd' and so on.

## Interpret the solution

Consider the time taken to evacuate residents from the building under the various assumptions. Describe the influence of the numerical changes in assumed values? What are the risks? What kind of robust estimate seems reasonable? Is it likely that the residents could exit safely in the event of a fire?

Estimate how much more quickly the building could be evacuated if both stairwells were safe and operational. Is it likely that residents could exit safely if both stairwells were functional?

## Evaluate the model

Consider how outcomes are affected by changes in numbers of people, speed of descent and delay times. Are these what you would expect? What does the model suggest are the most important influences on evacuation time? Is this consistent with intuition and experience?

## Report the solution

A simple modelling report will address the time question as it is written in the problem statement. It should contain all the above components of the modelling problem and its solution, including implications of further calculations using a range of different parameter choices. A more comprehensive report, going further than the question demands, would synthesise these data into a cohesive narrative, considering the implications for safety of the residents of the apartment building, and considering if possible, fire safety compliance standards set by the Australian building code. The precise structure of a report will depend on how the problem was expressed.

# Intermediate modelling problems for junior or middle secondary

The problems in this section are intermediate in demand. They are suitable for junior secondary to middle secondary students. Application of the modelling process in these problems involves more initiative and persistence. Mathematics is often initially absent, and needs to be introduced by the modeller.

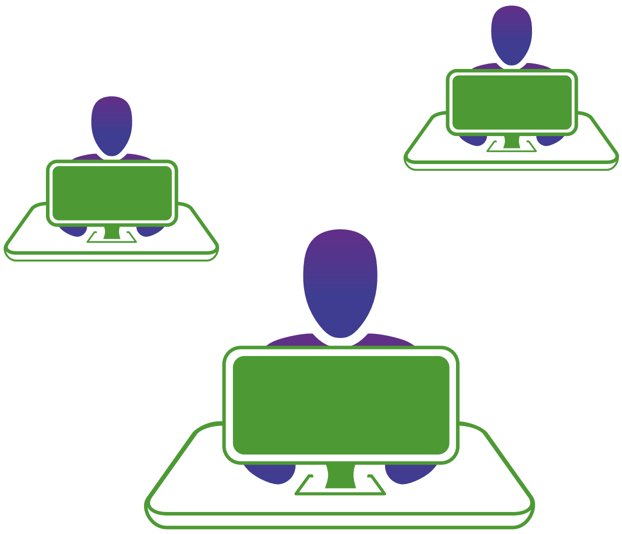
Classroom conditions for implementation will vary so it is important that all students have the opportunity to become modellers on an individual basis. For collaborative activity, students should work in groups of about four and learn to divide the work between them. They should start as a whole group and decide what needs to be done; what assumptions need to be made; what data are needed; which students will work on which sections; and how they will construct a summary report. To make such decisions they will need to be individually competent.

At this level more effort needs to be made on report writing. We suggest that students should be encouraged and helped to make summary reports for all of the models they tackle. Reports can be based around the content recorded as they progress through the different stages of the modelling process. It would also be useful for different groups to make verbal presentations of their report to the class. From time to time, a full report should be written and preferably assessed.

To help this communication process, we recommend that more writing be included in students' work on curriculum-oriented mathematics. Students should day-to-day be encouraged to add comments to their work to show how the next line follows or why they used a given approach. They should also present their answers in word form. For instance, instead of finishing with 'x = 5', they should be encouraged to write 'The largest possible root of the original equation is 5'.

In addition, in class, students should be helped to communicate their answers and methods to the whole class. So instead of giving a verbal answer of '5', they could say: 'I used the quadratic formula to solve the quadratic. The two solutions were -1 and 5. So the answer to the question "what is the bigger root of the quadratic?" is 5.'

This more verbal approach to mathematics will stand students in good stead at work and in social situations. If they are able to explain and discuss the way they have tackled a problem, mathematical or otherwise, they are more likely to be able to convince people of their solution and be more able to appreciate other people's arguments. The end result should be a better one for everyone.



# Waste not, want not

## Describe the real-world problem

> *On 16 February 2016, the Australian population reached 24 million people. Waste generation rates are a function of population growth, the level of urbanisation and per capita income and Australians now produce about 50 million tonnes of waste each year, averaging over 2 tonnes per person. There are more of us and we generate more waste per person, each year.*
>
> *In the period 1996–2015 our population rose by 28% but waste generation increased by 170%. Waste is growing at a compound growth rate of 7.8% /Year.*

Source: Mike Ritchie – Director, MRA Consulting Group https://blog.mraconsulting.com.au/2016/04/20/state-of-waste-2016-current-and-future-australian-trends/

Australia is one of the highest waste producers in the world, recently ranked in the top five waste producing nations, on a per person basis. In a year, we produce the equivalent of three million garbage trucks full of compacted rubbish. Every year the average Australian family produces enough rubbish to fill a three-bedroom house from floor to ceiling.

Some waste we are directly responsible for (e.g. household waste); other waste is generated on our behalf (e.g. from manufacturing processes, the building industry, road construction etc.). It is convenient to describe total waste generation in terms of amount per person. From the figures quoted, the total waste produced annually is currently 2.1 tonnes per head of population (approximately). Of this, household waste accounts for about one-seventh of the total.

On the positive side, recycling is growing at a faster rate and for the first time since 2005 we have seen a decline in tonnages of waste sent to landfill (in the most progressive states). We now recycle approximately 58% of all the waste we generate and landfill the rest.

Several modelling problems are suggested by this background information, in terms of implications for the future. We consider one such problem.

## Specify the mathematical problem

Find estimates in kilograms (or tonnes) for the total amount of waste produced in Australia over the next century.

## Formulate the mathematical model

### Assumptions

- Total waste produced in a year = population × waste ÷ person.
- Rates of increase that have applied over the past 20 years will continue into the future.
- Recycling or use of landfill do not enter at this point, as they are about managing waste that has been already produced. Here we are concerned with the production of total waste, to which re-cycling or landfill might later be applied.

### Parameter values

- Total waste is growing at a compound rate of 7.8% per year.
- Initial (2016) value for population is 24 000 000.
- Initial (2016) amount of waste per person is 2.1 tonnes per year.
- Estimate of amount of total waste produced in 2016
  = 24 000 000 × 2.1
  = 50 400 000 tonnes (assumption 1)

#### Model development

From the descriptions provided on the MRA website, rate of growth of waste follows a compound interest pattern.

#### Principle of compound growth

Amount next year ($A_1$) = amount this year ($A_0$) + added amount (interest).

$A_1 = A_0 + rA_0 = A_0 (1+r)$ where r = annual compounding (interest) rate.

Then $A_2 = A_1 + rA_1 = A_1 (1+r)$ gives the amount for the second year, and so on year by year.

Applying this principle to total waste generation (W):

$W_1 = W_0 (1+w)$

$W_2 = W_1 (1+w)$ and so on

Where $W_0$ =50 400 000; w =0.078.

Applying assumption (2) we now extend predictions into the future.

## Solve the mathematics

This is readily achieved using a spreadsheet.

| | A | B | C | D | E |
|---|---|---|---|---|---|
| **1** | Initial values | Waste growth rate | year | waste/year (tonne/yr) | Total waste (tonne) |
| **2** | Population (persons) | | 1 | = A3×A6 | =D2 |
| **3** | 24000000 | | 2 | = D2×(1+$B$6) | = (E2+D3) |
| **4** | | | 3 | Copy | Copy |
| **5** | Waste/ person/yr | Waste/yr | 4 | | |
| **6** | 2.1 | 0.078 | 5 | | |
| **"** | | | " | | |
| **"** | | | " | | |
| **101** | | | 100 | 8.5445E+10 | 1.1803E+12 |

## Interpret the solution

Column A contains the initial values of population, and of total waste produced per person in 2016.

Column B contains the assumed annual rate of increase of waste.

Column C contains the consecutive years over which the calculation runs.

Column D contains the total waste (in tonnes) produced on a yearly basis since 2016.

Column E contains the accumulated waste produced since 2016.

The last entry in column E estimates the number of tonnes of waste produced over the century – it represents an answer to the question originally posed.

## Evaluate the model

Writing out the estimate for total waste produced over the century looks like this:

From column E: Total waste produced over 100 years = 1 180 300 000 000 (over 1 trillion tonnes!)

This looks 'rather high' in practical terms, but we need something closer to home to relate to.

Column D tells us the amount of total waste generated on a yearly basis will be 85 440 000 000 tonnes per year in 100 years' time.

The background information says that over a 20-year period the population grew by 28%.

Assuming that population continues to grow at about the same rate it will not be too much in error to assume that in 100 years it will have grown to around 70 million.[7]

Dividing 85 440 000 000 by 70 million gives 1220 tonnes produced per person (approx) in the year 2115.

Noting that household waste formed about 1/7 of total waste, and assuming the proportion remains stable, this gives a quantity of about 174 tonnes of household rubbish per person for the year 2115.

On average this amounts to the production of about 0.48 tonnes (480 kg) per day for each person. That sounds like a lot of garbage (in more ways than one!)

After checking the mathematics, we are sceptical about the model predictions. What should be revisited?

[7] It isn't too much trouble to infer a compounding growth rate for population of 1.112 % p.a. from the given data, which when projected over 100 years from 2016 gives a population of about 73 million

### Model refinements

Assuming that the initial values are reliable, and population growth rates are fairly stable, we can identify growth rate for waste production 'w' as speculative. On the one hand we have assumed that the scale of growth observed over a 20-year period will continue for the next century. Is this reasonable? But beyond this, a closer inspection of the indicated growth rate (7.8% per year) given on the website is not consistent with the associated figure of a 170% increase in waste over the 20-year period – the implied rate is much less (about 3.95%).[8]

We can run the spreadsheet multiple times, using different values for 'w'.

Try halving the rate of growth of waste production. How does that look? What would it mean in real-life terms? Is it manageable? Similarly, for other choices.

Alternatively, set what are believed to be manageable levels of waste production and adjust 'w' in the spreadsheet to achieve these outcomes. This gives an idea of what efforts need to be made to contain the waste problem.

## Additional problem contexts to explore

### A. Recycling

How does Australia's recycling compare to the rest of the world?

Australians compete against the rest of the world in many ways and recycling is no exception. In some areas, such as newspaper recycling we have been world leaders for years, but we need to catch up in others. According to the MRA consulting website we now recycle 58% of all waste.

*What impact would this have on implications from the previous model? Make some assumptions about future improvements in recycling, and follow their implications by including them in a new model.*

### B. Implications for landfill

www.pc.gov.au/__data/assets/pdf_file/0016/21904/sub028.pdf

Australia has a strong dependence on landfill as a form of waste management, since the majority of waste that is not recycled or re-used in Australia is compacted and disposed of in landfills. The website contains much information about land filling including the following:

'… the depth of the landfill is assumed as 20m, the waste density is assumed as 750kg/m3 and the capacity is assumed to be 3 million tonnes.'

*Build a model that incorporates landfill as a means of disposing of waste that cannot be re-used or recycled, and consider future implications of the outcomes it produces.*

[8] This example shows the importance of evaluating models in terms of their real-world implications. In this case, evaluation led to the identification of an inconsistency in the data. In general, evaluating model outcomes means that the accuracy of claims made in the media can often be tested, which is another reason why the ability to do mathematical modelling is important for informed citizenry.

# Howzat!

Courtesy of *Teaching Mathematics and its Applications*

## Describe the real-world problem

The picture above was taken during a cricket Test Match between Australia and England at the Melbourne Cricket Ground. The photo shows the English batsman Colin Cowdrey trying to make his ground while taking a sharp single, against a run out attempt with Australian Wally Grout over the stumps. At the time there was no third umpire so the players had to rely on the square leg umpire for the decision. An umpire's judgment is still called on in all forms of cricket, except at the highest level where technology is available.

The umpire gave Cowdrey out. But would that have been the decision of a third umpire with digital technology to fall back on?

## Specify the mathematical problem

Use information from the photo to argue whether the 'out' decision was correct.

## Formulate the mathematical model

### Assumptions

Firstly, we need assumptions that will enable us to introduce mathematics into the problem, based on variables we identify as important.

We assume that the batsman achieves a speed (V) at the end of the run that is consistent with a batsman running a sharp single wearing batting gear.

We also assume (supported by the photo) that the bat has been grounded properly.

Most important is an assumption about the path of the bails. Did they fly horizontally on impact, or was there upward movement before descent? The most favourable case for the batsman is the former situation. (Students might discuss why.) This is the situation we will analyse first.

### Calculating time

The bail can be seen in the photograph as the dark mark roughly a quarter of the way down the left stump. An internet search gives the standard height of a cricket stump as 28 inches (71.1 cm). The vertical distance, 'h', that the bail has fallen can be estimated by direct measurement of the photograph.

The time (t) taken to reach this point after the stumps were broken can be estimated from the formula $h = (1/2)gt^2$.

(This is an application of the equation of motion

$h = ut + (1/2)\, at^2$

to the vertical movement of the bail, where a, the acceleration is $g = 9.8$ m/s$^2$, and $u = 0$ from the assumption of no upward movement of the bail.)

### Calculating velocity

Students will need to decide on a realistic estimate of the batsman's speed. How fast can a batsman with pads and bat run over a short distance? An internet search will give students a range of estimates to discuss. For example, finely tuned athletes run 100 metres at about 10 metres per second and 1500 metres at about 7 metres per second. Students will need to make assumptions about the effect of cricket pads on speed, the differences in the type of running between cricket and athletic sprinting and so on (a cricket pitch is only around 20 metres long, sprinters take some distance to accelerate from a standing start to full speed, etc.).

### Calculating distance

To estimate the distance that the bat is inside the batting crease as shown in the photo, we note that the distance on the photo between the two creases, the white line of the popping (batting) crease (in front of Grout) and the white line of the bowling crease (that passes through the base of the stumps) must be scaled to represent the actual value of 4 feet (as per the standard dimensions of a cricket pitch). A parallelogram can be drawn, using the shadow of the bat as a guide, that enables the distance of the bat inside the crease to be calculated as a fraction of this distance.

We can then estimate the distance (d) travelled by the batsman in this time at speed (V) using the formula d = Vt.

Now the distance 'd' travelled by Cowdrey since the wicket was broken can be compared with the distance 'r' his bat is inside the crease.  This decides the outcome. The umpire raised his finger – do you agree?

## Solve the mathematics

### Example data

It is best whenever possible for students to identify needed data and source data themselves as the modelling proceeds. They may find relevant sources alternative to those below.

We have drawn on references:
Laws of Cricket: https://www.lords.org/mcc/laws-of-cricket/
World Athletic Records: http://inglog.com/tools/world-records

For the purpose of this resource, an example data set is given below. Students conducting this exercise will likely find different data, and should experiment with the effect a range of values has on their model.

### Time since wicket was broken

$h = (1/2)gt^2$

therefore $t = (2h/g)^{\frac{1}{5}}$

$t = (0.3556/9.8)^{\frac{1}{2}}$

t = 0.19 seconds

### Distance travelled by batsman in this time

s = Vt
s = 7 × 0.19 = 1.33 m

### Position of batsman when wicket fell

r – s
1.10 – 1.33 = –0.23 m

## Interpret the solution

The batsman was out by about 23 centimetres.

Consider how outcomes are affected by changes in the values in the model.

## Evaluate the model

This can be extended. What if the bails went up in the air after the wickets were broken? Would that be better for the batsman or worse? If the bails went upward on impact the time for the bail to reach its position will be even longer than in this calculation. The batsman will be even more 'out', which also means that there is no need to explore this assumption further for purposes of deciding the question.

What if the batsman was running faster than the speed that was assumed? Our estimation of running speed introduced numerical data not provided in the photo. It can be reasonably argued (again suited to discussion) that the value chosen is likely to be an underestimate. Again, this confirms the decision as the faster the batsman is running the more trouble he is in.

| | | | |
|---|---|---|---|
| h | distance the bail has moved | 7 inches = 0.1778 m | estimated from photo |
| t | time for which the bail has been moving | $h = (1/2)gt^2$ | scientific formula |
| u | initial velocity of the bail | 0 | ball was not moving |
| a | acceleration of bail | $9.8\ m/s^2$ | gravity |
| v | velocity of batsman | 7 m/s | conservative value based on speed of professional athlete and accounting for effect of cricket padding |
| s | distance batsman has travelled since the wicket fell | s = Vt | scientific formula |
| r | distance of the bat inside crease at time the photo was taken | 90% of crease width = 0.9 × 4ft 0.9 × 1.2192 m ≈ 1.10 m | estimated from photo |
| | position of batsman when wicket fell, in relation to crease | r – s | solve for answer to problem |



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Further testing of the robustness of the original solution can be conducted by allowing the values of quantities subject to measurements made from the photo, or estimated, to vary (say by 10%). Does this make any difference to the outcome?

This is known as sensitivity testing and is important whenever the purpose of modelling is to determine a 'best' outcome. Here the outcome sought is a correct decision. 'Howzat' is an example of a prescriptive modelling problem - see the explanation of prescriptive modelling in the Introductory problem Adapt a Recipe.

Finally, as a matter of empirical interest, this problem has been used with groups of students at both secondary and tertiary level. The overwhelming response in every case is that the umpire got it right.

## Report the solution

The modelling report could contain all the above components of the modelling problem and its solution. The report should synthesise this data into a cohesive narrative, outlining decisions, assumptions and conclusions.

# How many trees make a newspaper?

## Describe the real-world problem

### How many trees go into your morning read?

According to the environmental conservation organisation Clean Up Australia, Australians' per capita consumption of paper products is high, and increasing every year. Australians consume more than four million tonnes of paper and cardboard annually. The environmental impact of paper consumption is significant. Although most used paper could be recycled, around half of it ends up in landfill.

## Specify the mathematical problem

Estimate how many trees are needed to provide a year's circulation of a major newspaper.

If the task is planned ahead of time, students can be asked to track daily papers for a week (or several weeks), and to bring newspapers to class when the problem is set.

## Formulate the mathematical model

This is an ideal team task. Form teams of four students who can then brainstorm, within their team, for a few minutes, a list of the data they will need to create a model, and the assumptions they will need to make. Outcomes can then be discussed and debated by sharing with the whole group. (Typically, not all necessary data and assumptions will be recognised at the start.)

Some examples are given below.

**Identifying data required**

Data required include:

- how much paper is in a newspaper, possibly considering
  - dimensions of a page
  - margin data, if it is decided to find how much paper is unused for actual print
  - daily page count or average pages over a week
  - weight of a page of newsprint
- how many newspapers are produced in a year, i.e., circulation (number of copies printed)
- weight of newsprint paper obtained from an average tree used for pulp; i.e., the number of trees required to produce a given weight of paper.

An ice-breaking very simple team activity is for each member to measure the dimensions of a page, both total and associated with print area. Averaging will give a measure to use in the model.

### Collecting data

How this proceeds depends on how much time is allotted to the task – for example spread over a week, or allotted a limited amount of class time? If the former, after identification of needed information, student teams can be assigned tasks of collecting their required data, such as:

- conducting internet research to find circulation figures of publications
- conducting internet research to identify relevant ways to convert quantities of paper into weight (kg). (If an electronic balance is available, weighing some newspaper is easiest)
- conducting internet research to estimate the amount of paper provided by a tree.

If time is limited, then websites can be provided to students. In that situation, the web links can be withheld until the students have identified the need for the associated information.

Many of the data requirements and ways of finding them will generate discussion. Weight of paper, for example, is a complicated measure. An internet search is likely to return some information about paper weight measured in standard base weight, using pounds (lb). A search will also find, however, that in many countries using the metric system, the relevant measure is grammage (grams per square metre, or g/m²).

In Australia, the weight of paper and paperboard is most commonly expressed as grammage, and it will be discovered that since the 1970s, the grammage of newsprint has decreased from a global standard of 52 g/m² to 48.8, 45 and 40 g/m². This is cheaper for newspaper producers, and increases the viability of forest resources. Students can consider why publishers might choose a certain weight of newsprint, and experiment with various weights in their model.

Additionally, there is no standard agreement on how many trees are required to create one paper's worth (or one pound, one kilogram, one tonne) of newsprint. Hence estimates are needed and these can vary with the source, because of variations in production processes.

The following websites are useful references:

Conservatree
*http://conservatree.org/learn/EnviroIssues/TreeStats.shtml*

Australian Science
*http://www.australianscience.com.au/environmental-science/paper-consumption-impact-in-australia/*

### Situational assumptions

- The number of pages is the same each week day, and each weekend, throughout the year.
- The pages have standard dimensions.
- The paper quality (thickness, weight) remains constant.
- The model excludes inserts and magazines, and assumes a 52 week year and additional copies not sold.

## Solve the mathematics

### Example data

For the purpose of this resource, an example data set is given below for a fictitious daily regional newspaper.

Given this information and the assumptions, weekly calculations can be completed and scaled up to give yearly estimates.

### Calculate paper usage

Per week
Single pages = (78 × 153 763 × 5) + (96 × 199 153)
+ (96 × 359 088) = 113 558 706
Single newsprint sheets (2 pages use 1 sheet of paper)
= 0.5 × pages = 56 779 353

| **Dimensions** | |
|---|---|
| Width (cm) | 29 |
| Height (cm) | 40 |
| Top margin (cm) | 1.8 |
| Bottom margin (cm) | 1.6 |
| Left margin (cm) | 1.8 |
| Right margin (cm) | 1.8 |
| **Average number of pages** | |
| Monday to Friday | 78 |
| Saturday and Sunday | 96 |
| **Circulation** | |
| Monday to Friday | 153 763 |
| Saturday | 199 153 |
| Sunday | 359 088 |
| **Weight** | |
| Assume a representative standard newsprint grammage | 45 g/m² |
| **Number of trees to weight of paper** | |
| Internet research informs assumption that one tonne of paper consumes approximately 12 full-grown trees (Conserveatree) | 12 trees per tonne |

Per year
Single newsprint sheets = 52 × 56 779 353 = 2 952 526 356
Area of sheet = 29 × 40 = 1160 sq cm = 0.116 sq m
Total area = 0.116 × 2 952 526 356 = 342 493 057sq m

### Convert area of newsprint to weight

For this example, assume a representative value for grammage 45 g/m$^2$. Other assumptions can be adopted to vary the calculations.

Standard newsprint grammage = 45 g/m$^2$ = 0.045 kg/m$^2$
Weight of newsprint = 0.045 × 342 493 057
= 15 412 187.6 kg = 15 412.2 tonne

### Convert weight of newsprint to trees

For this example, we assume a representative 12 trees per tonne of paper. Other assumptions can be adopted to vary the calculations.

1 tonne of newsprint uses 12 trees
15 412.2 × 12 = 184 946 trees

### Estimate area of trees needed

It is of interest to consider what this might look like in terms of forest area.

Internet research can be used to discover how many trees on average can be harvested/planted per hectare. This may result in different possible calculations.

State-based government departments of primary industries, for example, provide information on plantation management.

As an example calculation for this problem, using an average value of 1000 stems per hectare, 184 946 trees are needed to produce a year's worth of the example newspaper; that is, 184.9 hectares or 1.85 sq km.

### Generalise solution

Define symbols: page dimensions (l, w); number of pages weekly (n); weekly circulation (c); grammage (g); trees/tonne of newsprint (t).

Create a formula that can be used to estimate the number of trees needed for the annual production of any daily newspaper.

### Sensitivity testing

Using the formula for convenience, it is useful to vary the inputs from their values used in the example calculation and see how the outcome is affected. This tests the sensitivity of the result to changes in the inputs, and gives a sense of the robustness of estimates.

## Interpret the solution

In this problem, the building up of the solution has involved a consistent linking of all mathematical measures with their real-world meanings. So interpretation has followed mathematical calculations at all points.

## Evaluate the model

Firstly, the calculations should be checked to see that all essential variables have been built into the solution, and that arithmetic has been conducted accurately – noting that conversion between different systems of units may be involved. The generalisability of the model can be evaluated in terms of the formula developed for number of trees used, and its usefulness also in terms of insights gained from sensitivity testing.

### Refinement

The model can be refined by considering the impact of recycling. As an example for this problem, we have used information about recycling from Clean Up Australia *(http://www.cleanup.org.au/files/clean_up_australia_paper_cardboard_factsheet.pdf).*

> **Australia leads the world in newsprint recycling**
>
> *Australia's newsprint recycling rate is almost 10 per cent higher than in Europe, according to figures from the Publishers National Environment Bureau and media advocacy group The Newspaper Works. Annual figures over more than a decade show that Australia consistently recovers around 78 per cent of all newsprint. The average recycling rate for newsprint in Europe is 69 per cent. Australian newspapers only use tree pulp from plantation pine and use up to 40 per cent recycled materials; but newspaper and magazines are the most abundant forms of paper waste.*

How many trees are saved annually by using 40 per cent recycled materials in producing new newsprint, for the newspaper used in this example question?

0.40 × 184 946 = 74 000 (approximately)

## Report the solution

The modelling report could contain all the components of the modelling problem and its solution, as developed in preceding sections. The report should synthesise this data into a cohesive narrative, considering the implications of Australian plantation processes and use of recycled materials in producing new paper products.

# Bushwalking

## Describe the real-world problem

It is not uncommon for companions who enjoy bushwalking to differ in fitness and energy. On tracks that lead out and back they will often walk together for a time at the pace of the slower walker, until the slower walker indicates an intention to turn around and return to base.

The faster walker has the choice of following the same action, but alternatively may decide to carry on for a time at a faster pace before also returning. Especially if the opportunity to travel further and faster is appreciated, the faster walker will want to go as far as possible.

However, they will not want their companion to have to wait around too long at the end of the walk for them to return.

## Specify the mathematical problem

When the slower walker starts the return trip, for how much extra time should the faster walker travel on the outward path before turning for home, so that both will arrive at the starting point at the same time?

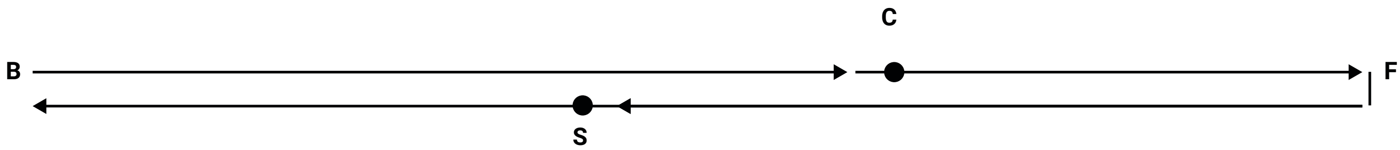
## Formulate the mathematical model

See the figure modelling the problem.

Let B = starting point and C = point that walkers reach together before separating.

Suppose that S is the point reached by slower walker (s) on the homeward path when faster walker (f) turns for home at F.

BF and FB are outward and return paths of the faster walker along the same track.



**Assumptions**

While together, both proceed at the speed of the slower walker. After parting, both walkers maintain their respective average walking speeds.

Let t = time taken to reach C while walking together at the speed V of the slower walker.

Slower walker (s) now starts to return while the faster walker (f) continues on as far as F at speed (kV) for an additional time T where k > 1.

For the walkers to arrive at B together:

time taken for f to cover the distance FB = time taken for s to cover the distance SB.

So FB/kV = SB/V

## Solve the mathematics

BC = Vt; CF = (kV)T; CS = VT

FB = BC + CF = Vt + kVT

SB = BC – CS = Vt – VT

Hence V(t + kT)/kV = V(t – T)/V

So (t + kT)/k = (t – T)

2kT = (k – 1)t

T = [(k – 1)/2k]t

T can now be found by substituting different values for t, V, and k.

(It is common for the formulation of a model to merge seamlessly into the mathematics of its solution.)

## Interpret the solution

It is sufficient to know the time from the start of the walk to the separation point (t), and the relative walking speeds (k) of the two individuals. For the first we need to remember to look at a watch at the start, and when point C is reached. (Note that spending some time at C before resuming the walk will not affect the outcome.) And relative speeds are easily estimated by comparing times taken to cover a chosen distance.

Firstly, check the mathematics for sensible outcomes.

k = 1 gives T = 0

Both walkers turn together if they walk at the same pace, as should happen.

Suppose walkers stay together for an hour.

t = 1 so T = (k – 1)/2k

Suppose k = 2: T = $\frac{1}{4}$ ('f' should continue on for 15 minutes). This seems a sensible figure.

Secondly, in terms of the real context, the outcome is amenable to direct checking. Try it out having estimated a value for k. This helps to underline that real-world problem solving cannot live entirely in a classroom.

## Evaluate the model

During evaluation of the solution to an original problem, a related problem at times suggests itself, and stimulates another application of the modelling cycle or parts of it. In this case we might consider that it is more realistic in practical terms for 'f' to aim to arrive back at base so that the slower walker doesn't have to wait 'too long' for the faster walker to return. That is, a modified modelling question is set in terms of a time window 'w', so that
0 < (difference in arrival times) < w.

Then in terms of formulating the model and solving the mathematics we have the following.
After the walkers separate, 'f' travels a distance (2CF + CB) at speed (kV) to reach base B.
Time for 'f' to reach base = (2kVT + Vt)/kV = (2kT + t)/k.

After the walkers separate, 's' returns to base B at speed V (same as outward trip).
Time for 's' to reach base = t.

So 'f' reaches base at a time
[(2kT + t)/k -t] = [2kT – (k – 1) [(k – 1)t + kw)]/2k t]/k
after 's' has arrived.

So we need 0 < [2kT – (k-1) t]/k < w

That is [2kT – (k-1) t]/k > 0 and [2kT – (k-1) t]/k < w

[2kT – (k – 1) t]/k > 0

gives T > t(k – 1)/2k

and [2kT – (k – 1)t]/k < w

gives T < [(k – 1)t + kw)]/2k

Hence t(k – 1)/2k < T < [(k – 1)t + kw)]/2k

For example, if w = 0.2 (12 minutes) and as before we consider the case where

k = 2, t = 1, then we obtain 0.25 < T < 0.35.

Interpreting, we have that if 'f' aims to have 's' wait no longer than 12 minutes at the starting point at the end of the walk, f should continue for about 15 to 20 minutes from the point where the walkers separate.

## Report the solution

The modelling report would contain all the above components of the modelling problem and its solution synthesised into a cohesive narrative. Additional working or explanations can be added when judged to enhance the product, with its completeness assessed in terms of, for example, the checklist on report writing.



ACER

# Farm dams I

## Describe the real-world problem

### *Farmers get their feet wet*

In a dry country like Australia, farm dams are part of the lifeblood of rural life. Knowing the volume of water at any time is important for planning the numbers and distribution of livestock, and estimating when the supply is likely to run out under drought conditions.

Methods of estimating the volume of a partly empty dam depend on interpreting physically observable signs. If depth markers are embedded when the dam is excavated, the volume can be estimated from the water level measured on the marker. If dams do not have markers, some other method is needed.

Dams lose water by evaporation from the water surface to the atmosphere. Annual average evaporation rates are estimated using data collected from locations throughout the country. They vary from about 100 cm in western Tasmania up to 400 cm in the desert regions of northern Australia. Values in Victoria vary from about 140 cm in the south to 180 cm in the north, according to the Bureau of Meteorology (http://www.bom.gov.au/watl/evaporation). Loss of water by seepage is negligible.

## Livestock water requirements

The table, containing data from a primary industries website, shows water requirements for a variety of farm animals. It contains information that farmers would know for their own stock, and is provided as a resource for this problem.

*Table 1 Livestock water requirements*

| Stock | Litres/animal/year |
|---|---|
| **Sheep** | |
| nursing ewes on dry feed | 3300 |
| fat lambs on dry pasture | 800 |
| mature sheep — dry pasture | 2500 |
| fat lambs — irrigated pasture | 400 |
| mature sheep — irrigated pasture | 1300 |
| **Cattle** | |
| dairy cows, dry | 16 000 |
| dairy cows, milking | 25 000 |
| beef cattle | 16 000 |
| calves | 8000 |

## Specify the mathematical problems

Find a method of estimating the amount of water in a partly empty dam.

We have made an assumption that the dimensions of the dam when empty (as excavated) are known, and that they are as shown in Figures 1 and 2. The dam (Figure 1) has been constructed by excavating a horizontal square base ABTW. Sides ABCD and WTSR are vertical, while the ends ADRW and BCST have been sloped back to ground level so that the distances AD and BC are equal to half the length of AB (lengths are not to scale). Livestock gain access to the water in the dam via the sloping ends.

Figure 2 shows the vertical cross-section ABCD. The side length of the bottom square ABTW is '2a', which is also the dam width. The depth of the dam when full is 'd', and 'h' represents the depth when it is partly empty. Sloping lengths AD and BC are of length 'a'. For this dam, a = 10 and d = 2 where the distances are in metres.

### Problem a: Model for dam volume

Find a way of estimating the volume of water in the dam when the (unknown) depth of water is 'h'.

### Problem b: Model for water loss over time

Suppose the dam is in northern Victoria. If the dam is full and no more rain falls, how many days it would take for the dam to dry up? Now suppose that a herd of 100 beef cattle is sustained by the dam. If the dam is full and no more rain falls, how long will it be before other arrangements need to be made for the stock?

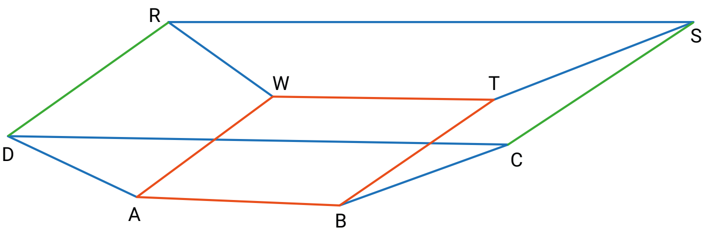



*Figure 1 Diagram of empty rectangular dam*

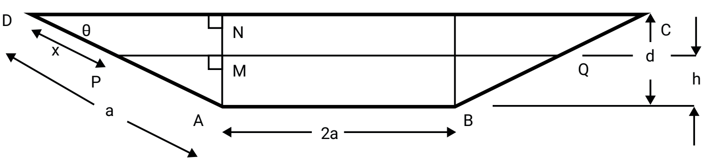



*Figure 2 Vertical section through rectangular dam*

## Formulate the mathematical model (a)

Discussion with students can be used to identify variables and measurements that will be needed to address the problem. Only data that can be accessed are useful, and the depth of water is not one of these. One readily obtainable piece of data is the distance the water surface has receded from its position when the dam is full — a distance obtained simply by stepping out the distance taken to walk from the edge of the dam, directly to the current water level. A key insight is the recognition that this distance ($x$) is both measurable, and a key input to the model.

### Assumptions

Dams with rectangular surface areas and with access for animals can be represented by trapezoidal volumes of various kinds. The dimensions of the dam when empty (as excavated) are known.

### Model for dam volume

We need to create a formula that enables the volume to be estimated when the distance DP = $x$ is known. ($x$ is the distance found by stepping out or measuring the distance the water has receded down the slope from its highest point when the dam is full.)

Volume of dam when full = area (ABCD) × width (2a)

In trapezium ABCD: length CD = AB + 2$x$ DN = 2a + 2acosθ where sin θ = d/a

Area of ABCD = $x$a (2 + cos θ) × d

Volume of dam (full) = 2 $a^2$d (2 + cos θ)

Given a = 10, d = 2, sin θ ≈ 0.2 (θ ≈ 11.5˚ and cos θ ≈ 0.98)

Volume of dam (full) = 1192$m^3$ (approx.) = 1.192 megalitres

To find volume of partly filled dam consider cross-section ABQP in Figure 2.

PM/MA = DN/NA (similar triangles) so PM/h = a cos θ/d and PM = ah cos θ/d.

Area (ABQP) = $\frac{1}{2}$ $x$h (4a + 2ah cos θ/d) so volume at height h is given by:

V(h) = area (ABQP) × 2a = 2$a^2$h[2 + (h/d) cos θ].

Hence V(h) = 200h(2 + 0.49h) – a quadratic in 'h'.

(Note as a check that when h = d = 2, V = 1192 as above.)

Now we need to express 'h' in terms of '$x$'.

By similar triangles DNA and PMA:
(a – $x$)/h = a/d so h = (d/a)(a – $x$) = 0.2(10 – $x$).

Substitution in the formula for V(h) gives:
V($x$) = 40(10 – $x$)(2.98 – 0.098$x$).

Check: When $x$ = 0, V = 1192 as above.

This formula can now be written more tidily as:

V($x$) = 3.92(10 – $x$) (30.4 – $x$) which rounds to a value of 1192 when $x$ = 0.

www.immchallenge.org.au

## Solve the mathematics (a)

Note that the constant checking of numerical values for consistency, during formulation, can also be considered part of the solution process. Volumes corresponding to different values of '$x$' calculated from the formula above are shown in Table 2 below. Volume columns show the respective percentages of the total capacity. Also shown (for comparison) are the values of the inaccessible depth measures (h) that correspond to the different selected values of the accessible measure ($x$).

Figure 3 is a graph of $V = 3.92(10 - x)(30.4 - x)$ for values of $x$ between 0 and 10.

It could be used, for example, as a chart pinned on the kitchen wall from which the volume of water remaining can be read, given only the distance ($x$) to the water line from the high water mark when the dam is full.

*Table 2 Dam volume and depth against distance dam water has receded from high water mark*

| $x$ (metres) | Volume (% of whole) | h (metres) |
|---|---|---|
| 0 | 100 | 2.0 |
| 1 | 87.0 | 1.8 |
| 2 | 74.7 | 1.6 |
| 3 | 63.1 | 1.4 |
| 4 | 52.1 | 1.2 |
| 5 | 41.8 | 1.0 |
| 6 | 32.1 | 0.8 |
| 7 | 23.1 | 0.6 |
| 8 | 14.7 | 0.4 |
| 9 | 7.0 | 0.2 |
| 10 | 0 | 0 |

## Formulate the mathematical model (b) for water loss over time

To estimate how long water in a dam will last, we need to make further assumptions – about evaporation.

### Assumptions

Water in the dam is reduced by evaporation and in providing for livestock. Seepage loss is negligible and can be ignored.

Internet research will typically show that the evaporation rate varies with temperature, wind speed, sunshine, and relative humidity.

It also varies throughout the year, but a rough daily evaporation rate (average) can be found in (cm or mm) by dividing the average annual value by 365 days. This will be sufficient accuracy for estimation purposes, although in practice there will be seasonal variations.

Using the value for northern Victoria given in the problem description, we obtain the average amount of evaporation per day $= 180/365 \approx 0.5$(cm).

This is the 'depth' of water that is lost across any exposed surface area in a day. The volume lost will vary with the surface area.

Calculations will overestimate the number of days that suitable water is available to animals. Near the end the dam will resemble a bog and the water undrinkable.

### Model for water loss over time

From Figure 2, the cross-sectional area when the depth of the dam is 'h' [A(h)] is a rectangle with length PQ and width 2a.

$A(h) = (2a + 2ah\cos\theta)/d \times 2a = 4a^2 (1 + h\cos\theta/d)$
$= 400(1 + 0.098h)$ when $a = 10$, $d = 2$.
When $a = 0$, $A = 400$ (area of the square base of the dam).

The area varies linearly with h, so the average cross-sectional area occurs when $h = 1$.
So average value of the exposed area of water surface $= 439.2\ m^2$, for which the daily evaporation loss would be $439.2 \times (0.5)/100 \approx 2.2\ m^3$.

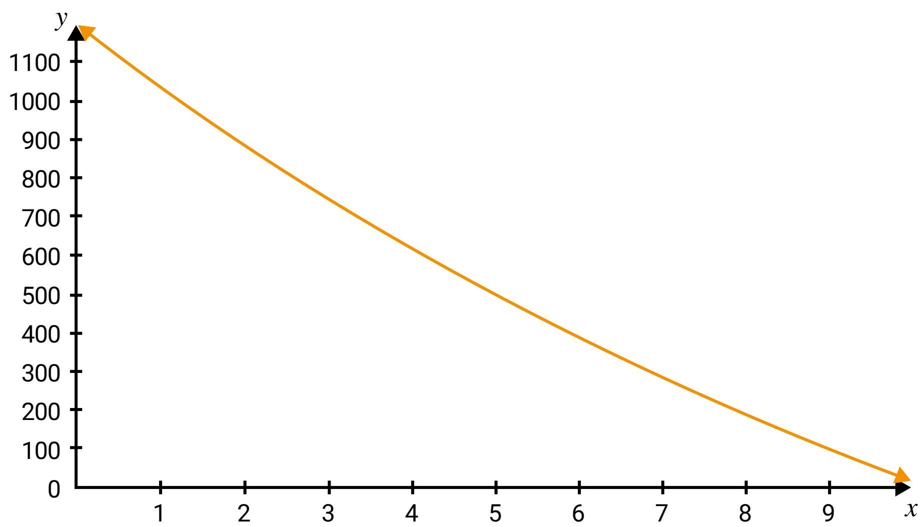



*Figure 3 Graph of volume against x for rectangular dam*

## Solve the mathematics (b) for water loss over time

If the dam is full and no more rain falls, we can estimate how many days it would take for the dam to dry up:
1192/2.2 ≈ 541 days (approximately 18 months' supply).

If the dam must sustain a herd of 100 beef cattle, then daily consumption of water by 100 beef cattle (from the example data):
16000 × 100/365 ≈ 4384 litres ≈ 4.38m$^3$.

Total average water loss per day from (consumption + evaporation) ≈ 6.58 m$^3$.

Number of days of water available before other arrangements must be made to maintain livestock ≈ 1192/6.58 ≈ 181 days (about 26 weeks or 6 months).

## Interpret the solution

Mathematical outcomes have been linked throughout to the dam structure, its volume and dimensions, and practical implications – as for example the meaning of the graph in Figure 3. This is typical of problems involving a variety of mathematical calculations. Their meaning within the problem needs to be interpreted and assessed as they arise, for testing numerical outcomes against the real situation will often identify errors in calculation that need to be addressed.

The formula obtained, translates the stepped out distance ($x$) into a corresponding value for volume that gives estimates of volume for any measured value of $x$. This would provide the basis for constructing a ready reckoner, or wall chart if desired.

The data provided, assumed a consistent evaporation rate based on an annual average. But the scenario posed — drought conditions — could be considered to be different from the average. How will the outcome vary, if different values are considered for the evaporation rate?

The data provided for livestock water requirements are annual averages. In a real-world scenario, are animals likely to need more water in drought conditions than in average weather? How will the outcome vary if different values are considered for the water consumption rate?

For water loss over time, noting the observation about boggy conditions when the dam is nearly empty, the figure obtained will overestimate the time drinkable water will be available. A safer estimate would be about 5 months.

## Evaluate the model

Apart from the checking of working for possible errors in mathematical calculations and/or in the application of technology, evaluation involves continuous checking against the needs of the problem context.

Has the solution provided a sufficiently good answer to the problem posed, or do we need further work?

Sometimes when the answer to the first question is 'yes' the first answer obtained suggests a deeper exploration that only becomes obvious from the initial modelling effort. This then stimulates another cycle of modelling with an amended purpose.

A different perspective on evaluation was reported by a teacher who used a version of this problem with her Year 10 class. An appreciative parent who happened to be a farmer told her that stepping down the bank was the method he and others used in estimating the amount of water in a dam.

## Report

The modelling report could contain all the above components of the solution of the problem. It should summarise and illustrate how the mathematical insights obtained advanced an understanding of the problem — even if this sometimes means that the solution attempt in its present state is in need of further development. All assumptions and choices of data values should be explained and justified.

In this case the report should develop as a systematic and cohesive narrative: considering the implications of drought conditions for livestock farmers; providing tools for farmers to use to estimate water supply, such as the data shown in Table 2 and the graph in Figure 3; and recommending a timeframe within which provisions should be made for alternative supplies for stock.

# Suggestions for student-generated modelling

At this stage, we encourage students to invent their own problems and to outline their approach to solutions. These problems should be related to things that students find interesting and important. Some topics with the potential to give rise to good modelling projects are indicated below.

### Music

- How many artists actually make any money from their music? How much is piracy costing performers?

### Politics

- What is the benefit of minor parties? How are they managed in other countries?

### Sport

- Is gambling good for the country? How well can winners be predicted?
- In archery, what is the optimum angle of elevation for the release of an arrow in the sport of archery so that the arrow hits the perfect bullseye? How does air resistance subsequently influence this optimum release angle?

### Games

- What games can be analysed to find winning strategies?

### Manmade disasters

- When will the Aral Sea in Uzbekistan dry up completely? Can it be saved?

### Catastrophic events

- How can you predict the severity of the damage of a tsunami from the Richter scale value based on energy?

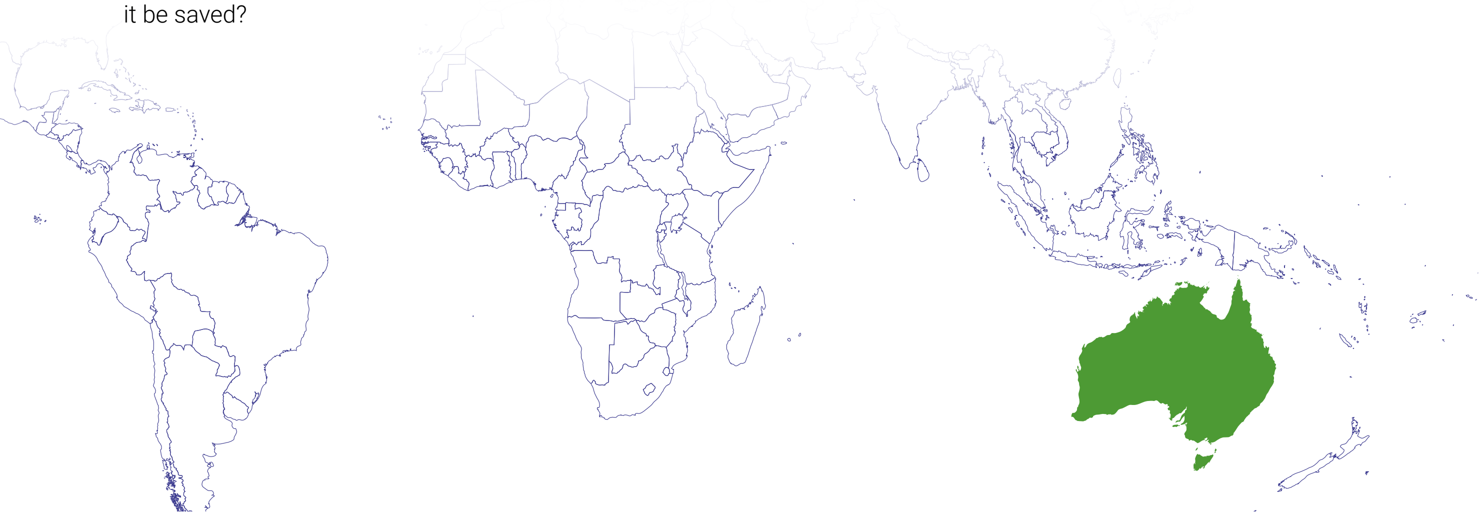
### Environmental problems

- Drought is causing water shortages in cities. How much water do we have left in the Hinze Dam? Will it cater for the current and future population of the Gold Coast?
- Analyse the spread of introduced biological control agents. What is the culling rate needed to stabilise the cane toad population in Australia?
- What do models tell us about climate change?

### Disease (epidemics and pandemics)

- What do the past and current trends in HIV/AIDS diagnosis and deaths suggest for the future number of people afflicted and their odds for surviving?
- Research Creutzfeldt–Jakob disease (sometimes described as the human form of 'mad cow' disease). The Chief Medical Officer in the United Kingdom has suggested fatalities from the disease could number in the hundreds of thousands. What are the rates of people who are infected but not showing symptoms, compared to fatalities?

A number of these were chosen and developed by students at a modelling challenge sponsored on the Gold Coast over several years by A B Paterson College and Griffith University.[9]



[9] See Galbraith, P., Stillman, G. & Brown, J. (2010). Turning ideas into modelling problems. In R. Lesh, P. Galbraith, C. Haines, & A. Hurford (Eds.), *Modeling students' modeling competencies* (pp. 133–144). New York: Springer.

# Senior modelling problems for middle or senior secondary

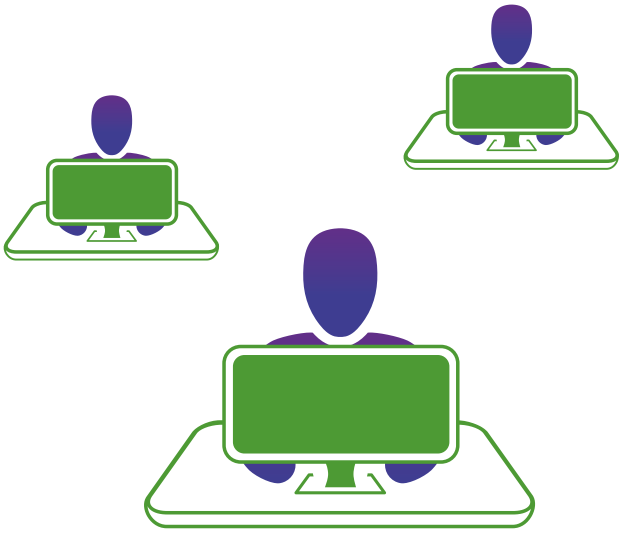
The problems at this level are designed to bridge from students' previous learning into the more substantial and complex demands of IM²C-style problems. These examples call upon initiative, persistence, decisions about the type of mathematics to apply, use of the web for data and solution methods, and how technology might be utilised. Generally, more advanced mathematical techniques will be needed. These problems contain scope for individuals and groups to exercise initiative and demonstrate attention to detail.

As in the intermediate-level examples, we suggest that, provided they have an understanding of the modelling process, students work in groups of about four and divide the work between them. They should start as a whole group and decide what needs to be done, what assumptions need to be made; what data are needed; which students will work on which sections; and how they will construct a summary report. However, all students should have an awareness of how the overall solution is proceeding, and feel free to contribute to any part of the process.

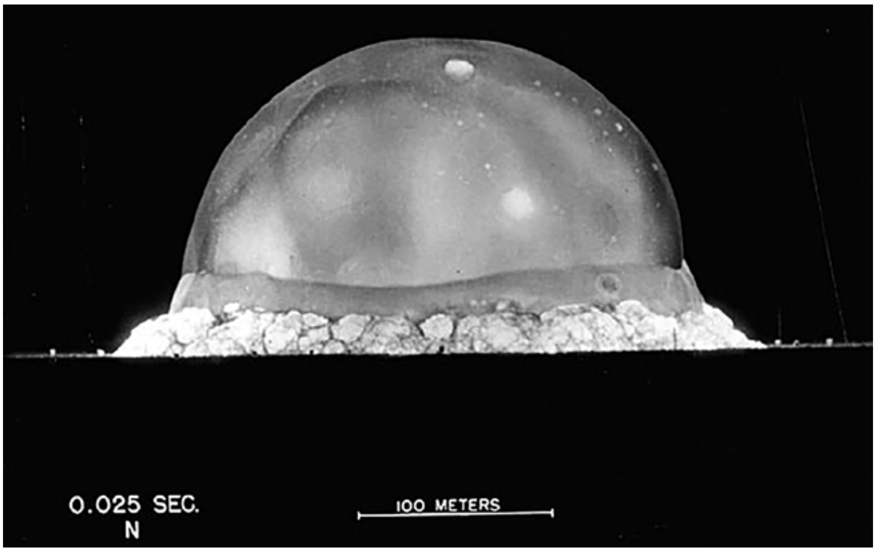
At this level, we would expect that more time be spent on report writing – again depending on total time availability. Students should write summary reports and deliver verbal results using their summaries. As much time should be spent on writing full reports as possible. It might be useful for student groups to assess each other's work. Preferably this should be a blind assessment.

The need to communicate is hopefully already a part of students' everyday classwork in maths. Students can develop skills in presenting both verbally to the class and in writing as part of their written answers. They should begin to treat their answers to assigned problems/exercises in some detail. Rather than giving an answer simply as 'V = 3.67', a more complete explanation can be provided, such as: 'I set up the volume, V, as an integral using the given function. Then I did the definite integration. The answer was 3.67. So I know that the volume of the dam is 3.67 gigalitres.'

This approach to mathematics ought to help the students' fluency in the subject (see the proficiency strand of the Australian curriculum: Mathematics) and their fluency in English. In real life, the first may not be as significant for them as the second. Being able to communicate is an important skill in life. This is especially true in the workplace, where increasingly, teams of people get together to solve problems. Because members of these teams are regularly drawn from various disciplines and backgrounds, they all need to able to communicate their different perspectives in order to maximise the value of their end product.



# Nuclear blast




Berlyn Brixner/Los Alamos National Labora

## Describe the real-world problem

The likely properties of the planned first atomic bomb were unknown until a trial detonation, the Trinity atomic test, was undertaken in the New Mexico desert in 1945.

Among the invited witnesses to the trial was a Cambridge Professor, Geoffrey Taylor, who had been an advisor to the Manhattan project team, the group responsible for developing the nuclear device. Simply witnessing the trial did not provide a measure of its strength.

Later, in 1947, photographs of the blast were made public in a variety of sources, including *Life* magazine. Taylor was browsing through one of these which contained a report of the test, together with photos of the expanding blast wave, taken over a succession of small time intervals. Armed with a series of photos, he devised a question and conducted modelling to estimate the energy released in the blast.

Taylor's approach to the problem has been discussed in various sources, including by University of Cambridge Professor Timothy Pedley in the journal *Mathematics Today* in 2005. The paper describes (with illustrations) how to apply mathematics to real-world problems, including a simplified working of the nuclear blast problem.

Sufficient information is provided to enable the investigation to be illustrated through a modelling approach.

## Specify the mathematical problem

Estimate the energy released in the bomb blast.

## Formulate the mathematical model

This problem illustrates two significant aspects about modelling activity. Firstly it illustrates an important attribute of a modeller – the identification by modellers of model-rich situations in the first place. Curricular goals for students to use their mathematics to solve problems of personal interest, in work contexts, and as productive citizens require this ability. (The International Mathematics Modeling Challenge program does not explicitly address this aspect, as by its nature problems are specified as the starting point.) Secondly, this problem affirms the use of a cyclic modelling process by a professional modeller.

The fundamental assumption is that the radius of the spherical blast wave (R) depends on a product of three factors: the time elapsed since the explosion (t), the instantaneous energy released (E), and the density of air (ρ). Thus $R = Ct^aE^b\rho^c$, where C is a dimensionless constant.

(These assumptions are taken as a given for this development. Their plausibility can be discussed if desired.)

## Solve the mathematics

Taylor then proceeded to apply dimensional analysis as elaborated below, in order to find suitable values for the indices a, b and c, and to use the results to estimate the amount of energy released.

Using the standard notation used for dimensions dim R = [R] etc, to express quantities in terms of the fundamental dimensions mass (M), length (L), and time (T) we have:

$[R] = L$, $[t] = T$, $[E] = ML^2 T^{-2}$, and $[\rho] = ML^{-3}$

Thus dimensionally, using the formula above, we need:
$L^1 = M^{(b + c)} L^{(2b - 3c)} T^{(a - 2b)}$

Equating dimensions on both sides of the equal sign we need : b + c = 0; 2b − 3c = 1; a − 2b = 0

hence $a = \frac{2}{5}$, $b = \frac{1}{5}$, and $c = -\frac{1}{5}$.

So $R = C\,(Et^2/\rho)^{\frac{1}{5}}$

(A value of $C \approx 1$ was assigned on the basis of knowledge of blast activity, and hence $E = \rho R^5/t^2$.)

The above photograph of the blast contains a scale representing 100 m, and a label indicating that it was taken at t = 0.025 (s). Expanding the photo from the link, taking measurements, and using the given scale to estimate the radius suggests a value for R of about 132 m.

Noting that density of air is 1.2 kg/m$^3$ and substituting in the above formula gives a value for E of about $7.7 \times 10^{13}$ joule.

## Interpret the solution

The standard measure of strength of bomb blasts is given as a comparison with the yield of the explosive material TNT, trinitrotoluene. Internet research indicates that the energy released by 1 metric tonne of TNT is equivalent to about 4.148 gigajoules, where 1 gigajoule equals 1 billion joules. Check that the value of E for the Trinity blast calculation (using the single diagram above) converts to an energy equivalent of about 18.4 kilotonnes of TNT.

## Evaluate the model

The method used by Taylor was slightly different from the above which is based on measurement and calculation from one photograph. Taylor used data from the series of photographs to plot the graph of log R against log t.

(Note that $R = (Et^2/\rho)^{\frac{1}{5}}$ can be written as
$\log R = \log (E/\rho^{\frac{1}{5}}) + 2 \log t$, so that the graph of log R against log t is linear. The value of E can then be robustly estimated from the intercept on the vertical axis). This is illustrated briefly in the Pedley article.

An internet search will identify many sources that discuss aspects of the Taylor approach. Several of these provide a series of photographs of the blast wave at successive time intervals. These can be used to give separate estimates of the energy released (as in the calculation above), or together to generate a log-log plot as used by Taylor. The sources do need vetting before being given the students, as some include unnecessary complications.

A useful source (Codoban, 2004) is located at
http://www.atmosp.physics.utoronto.ca/people/codoban/PHY138/Mechanics/dimensional.pdf

## Report the solution

Despite post-war national security issues, Taylor published his results showing the Trinity atomic bomb had a power equivalent of about 17 kilotons of TNT. The US Army was annoyed – the information was supposed to be classified!

## References


Codoban, S. (Ed.) (2004). *Estimate of the energy released in the first atomic bomb explosion* [tutorial notes for University of Toronto subject Physics for the Life Sciences]. http://www.atmosp.physics.utoronto.ca/people/codoban/PHY138/Mechanics/dimensional.pdf

Pedley, T.J. (2005). Applying Mathematics. *Mathematics Today*, 41(3), 79–83.

Taylor, G.I. (1950). The formation of a blast wave by a very intense explosion. II. The atomic explosion of 1945. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences, 201*(1065), 175–86. http://www3.nd.edu/~powers/ame.90931/taylor.blast.wave.II.pdf

# Population growth

## Describe the real-world problem

### Australia's population to hit 23 million

Tuesday, 23 April, 2013.

Australia's population will reach 23 million people overnight, and is on track to surpass 40 million within 40 years.

Projections build on last year's known population, and take into account fertility, life expectancy and immigration figures. Figures from the Australian Bureau of Statistics show that around 180 000 people move to Australia each year.

The ABS estimates that there is a birth every one minute and 44 seconds, a death every three minutes and 32 seconds, and a new migrant arriving every two minutes and 19 seconds.

That means our population increases by one person every minute and 23 seconds – more than 1000 people per day.

## Specify the mathematical problem

**Problem a: Daily population growth**

Simple problem. Check and comment on the claim that 'our population increases by one person every minute and 23 seconds – more than 1000 people per day.'

This amounts to evaluating claims made by others on the basis of someone else's model. This is itself an important activity to recognise and undertake when appropriate.

**Problem b: Population growth over 40 years**

Advanced problem: Is it likely that Australia's population will reach 40 million in 40 years?

## Formulate the mathematical model (a) for daily growth

Increase in persons per day = no of births per day – no of deaths per day + no of net migrants per day.

## Solve the mathematics (a) for daily growth

As on 23 April 2013:

no of births per day = (24 × 60 × 60)/104 = 830.77
(one birth every 104 sec)

no of deaths per day = (24 × 60 × 60)/212 = 407.55
(one death every 212 sec)

no of (net) migrants per day = (24 × 60 × 60)/139 = 621.58
(one migrant arrives every 139 sec)

So, increase in persons per day = 830.77 − 407.55 + 621.58
= 1044.8 (1045 persons)

Time interval between arrivals = (24 × 60 × 60)/1045
= 82.68 sec (1 min 23 sec approx)

## Interpret the solution (a) for daily growth

The calculations verify the claims that on this day the population increases by one every 1 min 23 sec, which is more than 1000 persons per day

## Evaluate the model (a) for daily growth

The model is built in terms of births, deaths, and net immigration, and gives results precisely in line with the claims. We can be confident it is a good model for the purpose it was created.

However, we can point to reservations about its wider use. The model uses data for births, deaths, and net immigration that are applicable on a particular day. For population predictions long-term, we need annual estimates of these quantities. For example, a migration rate of 621 per day (the figure that applies on 23 April 2013) if used to calculate an annual figure gives a value over 226 000 (many more than the average of 180 000, given by the ABS).

The challenge of estimating population size into the future is considered in the next problem.

## Formulate the mathematical model (b) for growth over 40 years

The advanced problem asks: Is it likely that Australia's population will reach 40 million in 40 years?

Given that births, deaths, and net migration have been established as the key variables, the emphasis changes to designing a model for long-term population forecasts – typically expressed in terms of *years*.

It is useful to summarise what we know from the data given in the report (see the table). The last column generalises the calculations to express the *yearly* change in a population (P) in terms of the respective contributions from births, deaths, and migration.

The calculations assume that the data given for April 23 apply across the year, which seems to be a suggestion within the reporting. This is a point for discussion in its own right.

### Assumptions

For our first model we will use the values given in the report, and included in the table. This assumes that the fractional birth rate, the fractional death rate, and the rate of migration remain the same over the time scale of the model.

Note that in published statistics, migration rate refers to the net rate of increase through migration.

These assumptions deserve discussion, as they provide a means of making progress, but may also be a source of reservation in accepting predictions. They will be revisited when evaluating model outcomes.

| **Population in 2013 = 23 000 000** | | | | | **Population = P** |
|---|---|---|---|---|---|
| **Births** | one per 104 seconds | births per day<br>= (60 × 60 × 24) ÷ 104<br>= 830.77 | births per year<br>= 830.77 × 365.25<br>= 303 439 | Fractional birth rate (b)<br>= births per year ÷ total pop<br>= 303 439 ÷ 23 000 000<br>= 0.0132 per year ($yr^{-1}$) | births per year<br>= Pb |
| **Deaths** | one per 212 seconds | deaths per day<br>= (60 × 60 × 24) ÷ 212<br>= 407.55 | deaths per year<br>= 407.55 × 365.25<br>= 148 856 | Fractional death rate (d)<br>= deaths per year ÷ total pop<br>= 148 856 ÷ 23 000 000<br>= 0.00647 per year ($yr^{-1}$) | Deaths per year<br>= Pd |
| **Net migration** | one per 139 seconds | migration per day<br>= (60 × 60 × 24) ÷ 139<br>= 621.58 | migration per year<br>= 621.58 × 365.25<br>= 227 032 | Migration numbers (m)<br>= 227 032 persons/year | Migrants per year<br>= M/365.25<br>where<br>M = total migrants in a year |
| Note that net migration is independent of current population, while the number of annual births (Pb) and annual deaths (Pd) is directly influenced by it. | | | | | |

Real-world material relevant to both modelling and discussion can be obtained from internet sources. Students will often do this on their own initiative, or can be prompted if relevant. (It is important that such activity remains focused on the modelling task – for example, is related to clarifying assumptions – and does not become a diversion).

## Solve the mathematics (b) for growth over 40 years

For this first model, we use the data provided in the stimulus text, and summarised in the table. There are a number of ways this problem can be approached mathematically, and three different approaches are shown.

The spreadsheet solution can be applied by anyone with spreadsheet competence. The other solutions require knowledge of geometric series and calculus respectively, which locates them within senior mathematics.

Let $P_0$ = initial population (in 2013)

Let $P_n$ = population in year n

Let r = natural population change rate
= birth rate (b) – death rate (d) i.e. (b – d)

Let M = average net annual immigration intake

**To solve by spreadsheet**

$P_0 = 23\,000\,000$; b = 0.0132; d = 0.00647; r = b – d = 0.00673; M = 227 032

$P_1 = P_0 + rP_0 + M = P_0 (1 + r) + M = P_0 R + M$ (where R = 1 + r)

$P_2 = P_1 R + M$ etc

•

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$P_{40} = 40\,459\,252$

**To solve by geometric series**

Proceeding as above (annual increments)

$P_1 = P_0R + M$

$P_2 = P_1R + M = P_0R^2 + M (R + 1)$

•

•

$P_n = P_0R^n + M (R^{n-1} + \dots R^2 + R + 1)$

$P_n = P_0R^n + M (R^{n-1})/R - 1$

$P_{40} = 40\,459\,252$ (annual increments)

$P_{40} = 40\,523\,429$ (monthly increments)

$P_{40} = 40\,523\,960$ (daily increments)

**To solve by calculus**

$\delta P \approx Pr\,\delta t + M\,\delta t$

$= (Pr + M)\,\delta t$ where r = b – d.

In the limit

$dP/dt = Pr + M$

$\ln(Pr + M) = rt + \text{constant}$, leading to

$P(t) = P_0e^{rt} + M(e^{rt} - 1)/r$ where $P(0) = P_0$

$P_{40} = 40\,526\,191$

## Interpret the solution (b) for growth over 40 years

All methods confirm a predicted population of around 40 million in 40 years' time.

## Evaluate the model (b) for growth over 40 years

Appropriate mathematical growth processes have been applied, and similar results obtained using three different methods. It seems reasonable to trust the methods.

What about the assumptions leading to the values used for b, d, and M which were all based on their values on a particular day: April 23, 2013?

It is easy for death rates to have a 'spike' – for example during 'flu epidemics, or for some reason to be lower than average during a short interval. For predictions, we need stable average values. If members of a species have an average life time of 10 years, then on average 1/10 of a population (the fractional death rate) will die each year. Generalising, an average lifetime of 'L' means that on average a fraction '1/L' of the population die each year. In the fifth column of the table we see that d = 0.00647 $yr^{-1}$, which implies an average life time of 154.6 years.

Clearly the value on April 23 2013 was not typical and we need a more representative value.

Websites such as the Australian Institute of Health and Welfare give figures for life expectancy for Australians (http://www.aihw.gov.au/deaths/life-expectancy ).

Life expectancy is not quite the same as average lifetime but is a close approximation (an interesting point of discussion if time is available). If the average life time in 2013 was 81.5 years, the average fractional death rate = 1/81.5 = 0.0122699 $yr^{-1}$.

Also, the net immigration rate is quoted officially at about 180 000 per year, which is substantially less than the figure implied by the specific value that applied on the special date.

Leaving the value of 'b' alone, which is essentially determined by personal family planning decisions, and adjusting the values of 'd' and 'M' as above, show that using the above three methods of calculation, leads to a value of $P_{40}$ = 31 200 000 (approximately).

This is substantially less than the previous estimate. There are implications for jobs, housing, health provision, education… if forecasts are relied upon. This adds a social context around the problem.

### Refinement (advanced)

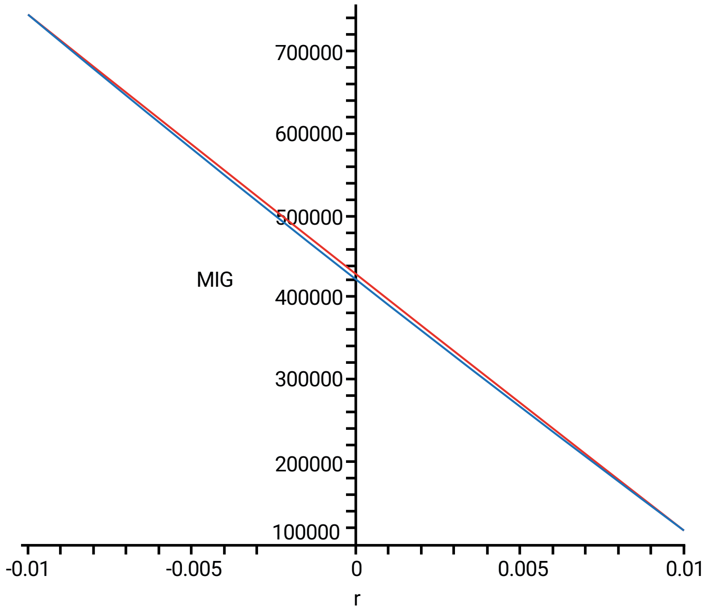
Over time, the natural growth rate (births – deaths) is monitored and published as statistics. Knowing these, immigration can be used as a policy tool to address population issues. We can use this idea to pose a different question.

Suppose a population of 40 million is the goal for 2053 (40 years from 2013). What combination of natural growth rates and immigration rates would achieve this?

For this senior level refinement using $P = P_0e^{rt} + M(e^{rt} - 1)/r$ from previous work we need

$40\ 000\ 000 = 23\ 000\ 000\ e^{40r} + (e^{40r} - 1)I/r$

This problem requires facility with CAS technology such as Maple or Mathematica. Using the former and noting that MIG = M the graph is shown on the axis below (blue in online format).




An almost identical graph of a linear approximation has equation M = 430 652 – 31 500 000r (red). Now feasible solutions can be considered by noting the real world meanings of r and M. For example if r = 0 (so called replacement rate when deaths are just matched by births) then M = 430 652, leading to a population of 40 226 080 in 2053. Is this realistic? Allowing ‘r’ and ‘M’ to vary within the 40 year period provides for considering implications of changes in family planning decisions (personal) and immigration policy (national).

## Report

The modelling report should contain all the components of the modelling problems, their solutions, interpretation and evaluation. The report could provide a cohesive narrative, including some discussion of implications of Australian population growth for policy, society and economy around matters such as jobs, housing, health, and education. When forecasts are astray where does responsibility lie? With demographers? With data problems? Has the media been creative with facts or interpretations?

# Fifteen-forty

## Describe the real-world problem

### Results: The Championships, Wimbledon 2015

Ladies' singles: *Serena Williams* def *Garbine Muguruza*: 6–4, 6–4

Gentlemen's singles: *Novak Djokovic* def *Roger Federer*: 7–6 (7–1), 6–7 (10–12), 6–4, 6–3

Some years ago a tennis commentator made the following remark during a Grand Slam tournament: 'A top male player has a fifty-fifty chance of winning a game from 15–40 on serve.'

## Specify the mathematical problem

Evaluate this statement. That is: what probability can we assign to the outcome of winning a service game from a score of 15–40?

## Formulate the mathematical model

**Data**

Students can be given the data table below or tasked with collecting the required data. The official Wimbledon website contains many years' worth of results, including the most up-to-date live scores. http://www.wimbledon.com/en_GB/scores.

Other Grand Slam tournament websites also contain similar data.

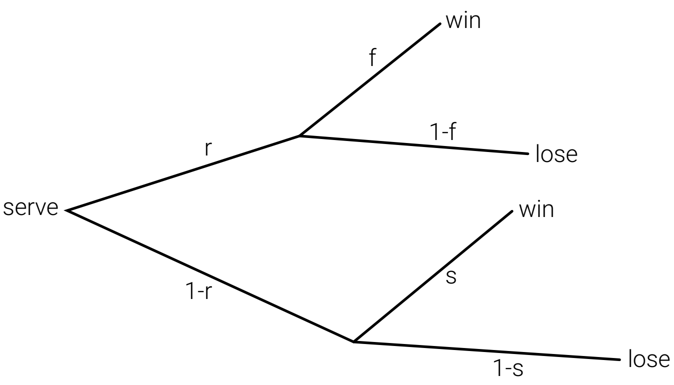
**Finals match data**

| Wimbledon 2015 | *Williams* | *Muguruza* | | *Djokovic* | *Federer* |
|---|---|---|---|---|---|
| 1st serve in (r) | 37 of 68<br>54% | 43 of 61<br>70% | | 95 of 145<br>66% | 94 of 141<br>67% |
| Winning 1st serve (f) | 29 of 37<br>78% | 23 of 43<br>53% | | 70 of 95<br>74% | 70 of 94<br>74% |
| Winning 2nd serve (s) | 11 of 31<br>35% | 6 of 18<br>33% | | 30 of 50<br>60% | 23 of 47<br>49% |

**Situational assumptions**

Given the long run frequency of serving outcomes we make the following assumptions:

- The probability of the outcome of any particular serve is estimated as the long run proportion for the outcome of serves as given in the table.
- The outcome of any point is independent of the result of the previous point. (Note that the reliability of these assumptions should increase with the quality of the players.)
- There are two mutually exclusive outcomes for each point (win or lose).




Outcomes of a service point

To win a point:
either the first serve is in (with probability given by r) and the service point is won (with probability f)
or the first serve is out (with probability 1 – r) and the second serve is won (with probability s).

Hence, as seen in the figure using the notation from the table, the probability of winning a service point (p) is given by
$p = rf + (1 - r)s$

## Solve the mathematics

Using the data in the table, we obtain:
p (*Djokovic*) = $(0.66)(0.74) + (0.34)(0.60) = 0.69$
p (*Federer*) = $(0.67)(0.74) + (0.33)(0.49) = 0.66$

Note that $q = 1 - p$ is the probability of losing a service point, and that the probability of winning a point as receiver is the complement of the opponent's probability of winning as server.

Let Pr(G) be the probability that a player wins a service game from 15–40.

The server must win the next two points and then win from deuce.

So our required probability is $\Pr(G) = p^2 \times \Pr(D)$, where Pr(D) is the probability of winning a service game from deuce.

To win from deuce the server must win the next two points (probability $p^2$)
or return to deuce and win from deuce.

To return to deuce the server must win the first point and lose the second (pq) or vice-versa (qp).

Hence $\Pr(D) = p^2 + 2pq \times \Pr(D)$

Hence $\Pr(D) = p^2/(1 - 2pq) = p^2/(1 - 2p + 2p^2)$ since $q = 1 - p$.

So $\Pr(G) = p^4/(1 - 2p + 2p^2)$

Check:
If $p = 0$, $\Pr(G) = 0$ and if $p = 1$, $\Pr(G) = 1$
as should occur.

## Interpret the solution

If we take the case of *Federer* then $p = 0.66$

Substituting this value for p gives $\Pr(G) = 0.34$
That is, a 34 per cent chance of winning the game.

Verify that for *Djokovic* the figure is around 40 per cent.

We can find the value of 'p' that will meet the commentator's criterion as follows.

We need to solve $p^4/(1 - 2p + 2p^2) = 0.5$
That is,
$2p^4 - 2p^2 + 2p - 1 = 0$ which we can solve with the help of Computer Algebra System (CAS) software.
$p \approx 0.75$ is the single positive root

Check:
When $p = 0.75$, $\Pr(G) = 0.506$ (approximately 50 per cent chance of winning).

It seems the commentator was a bit over optimistic!

But, Wimbledon finals is the very best playing against the very best. If the opponent is a lesser player (which happens most of the time) then the opinion might be closer to the mark.

The Wimbledon website contains complete statistics for all seven matches played by finalists. To obtain them click on the names of players.

The collated data for *Djokovic* gives outcomes for 728 service points across seven opponents.

Using the total data for the same variables shown in the table we calculate:
$r = 0.707$ (0.71)
$f = 0.77$
$s = 0.636$ (0.64)
giving $p = 0.731$ (0.73)

With $p = 0.73$ we obtain $\Pr(G) = 0.469$ (around 47 per cent chance of winning).

So the estimate is looking quite a reasonable one, especially when we consider that many opponents on the circuit will be lesser players than those accepted for a Grand Slam tournament.

## Evaluate the model

We can also take a statistical look at outcomes.

Assume that the choice of the seven opponents in the draw is sufficiently random to support the estimation of confidence intervals. (The sample is probably best regarded as representative rather than random.)

Consider the 728 service points from Wimbledon 2015 as a sample from the population of many thousands that a quality player, such as a Grand Slam winner, is involved with while at the top of his or her form over a period of years.



ACER

Considering the probability of winning a point as given by the proportion of service success, then the standard error of the sampling distribution of proportions is given by $(\sqrt{(p^*(1-p^*)/n)}$ where $p^*$ is the value (0.73) from the sample ($n = 728$).

Then with (approximately) 95 per cent confidence we estimate the population value of p to lie in the interval:
$p^* - 2(\sqrt{(P^*(1 - p^*)/n)} \le p \le p^* + (2\sqrt{(P^*(1 - p^*)/n)}$

That is $0.698 \le p \le 0.764$ which translates to
0.409 (41%) ≤ Pr(G) ≤ 0. 531 (53%)

Given that some opponents will not be of Grand Slam quality, we might argue that this outcome is probably still conservative at the upper end.

It is interesting that the original Wimbledon data generated a p-value (0.69) that lay just outside the lower 95 per cent confidence limit. It is probably no surprise that such an extreme occurrence took place with Roger Federer as the opponent.

The first evaluation suggested further avenues to explore, which were not envisaged at the start. This is an illustration of how interim results can often stimulate further modelling activity in which aspects of the modelling cycle are re-activated, sometimes a number of times.

## Report the solution

This analysis has focused around Novak Djokovic as a case study. There is opportunity to test the commentator's claim using a selection of top players, and ideally this should be pursued.

For example, what are the corresponding outcomes for top women players? A similar approach can be used to investigate parallel outcomes for women players, using data from the website.

Perhaps Serena Williams is just about the best of all time? Use available data to compare her serving performance with that of other top women players.

Further problems could also be explored. For example, what are reasonable probabilities of winning a game from each of the positions from 0–40 to 40 –0?

Using the same notation, show that the probability of winning a game from 40–0 can be expressed as:
$Pr(G) = p(1+ q + q^2) + q^3 Pr(D)$
$Pr(G) = 0.998$ when $p = 0.73$
and so on.

# Farm dams II

## Describe the real-world problem

### *Farmers get their feet wet*

In a dry country like Australia, farm dams are part of the lifeblood of rural life. Knowing the volume of water at any time is important for planning the numbers and distribution of livestock, and estimating when the supply is likely to run out under drought conditions.

Methods of estimating the volume of a partly empty dam depend on interpreting physically observable signs. If depth markers are embedded when the dam is excavated, the volume can be estimated from the water level measured on the marker. If dams do not have markers, some other method is needed.

Dams lose water by evaporation from the water surface to the atmosphere. Annual average evaporation rates are estimated using data collected from locations throughout the country. They vary from about 100 cm in western Tasmania up to 400 cm in the desert regions of northern Australia. Values in Victoria vary from about 140 cm in the south to 180 cm in the north, according to the Bureau of Meteorology (http://www.bom.gov.au/watl/evaporation). Loss of water by seepage is negligible.

## Livestock water requirements

The table, containing data from a primary industries website, shows water requirements for a variety of farm animals. It contains information that farmers would know for their own stock, and is provided as a resource for this problem.

*Table 1 Livestock water requirements*

| Stock | Litres/animal/year |
|---|---|
| **Sheep** | |
| nursing ewes on dry feed | 3300 |
| fat lambs on dry pasture | 800 |
| mature sheep — dry pasture | 2500 |
| fat lambs — irrigated pasture | 400 |
| mature sheep — irrigated pasture | 1300 |
| **Cattle** | |
| dairy cows, dry | 16 000 |
| dairy cows, milking | 25 000 |
| beef cattle | 16 000 |
| calves | 8000 |

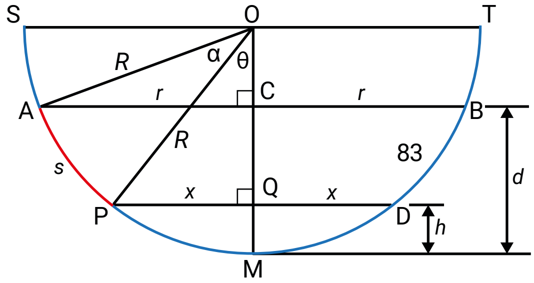



*Figure 1 Cross-sectional diagram of a partly full dam*

## Specify the mathematical problems

The dam (Figure 1) has been excavated in the shape of an inverted spherical cap AMB – so that its cross-section is circular whatever the depth of water, and stock can drink from any point on its perimeter. The diameter of the dam when full (AB) is 30 metres, and its depth when full is 2 metres. The figure shows cross-sections of the dam, and of the hemisphere SMT of which it is a part. PMD is a representative section of the partly filled dam of radius 'x' and depth 'h'.

### Problem a: Model for dam volume

Find a way of estimating the volume of water in the dam when the (unknown) depth of water is 'h'.

### Problem b: Model for water loss over time

Suppose the dam is in central Victoria. If the dam is full and no more rain falls, how many days it would take for the dam to dry up?

Now suppose that a flock of 300 mature sheep on dry pasture is sustained by the dam. If the dam is full and no more rain falls, how long will it be before other arrangements need to be made for the stock?

## Formulate the mathematical model (a) for dam volume

Discussion with students can be used to identify variables and measurements that will be needed to address the problem. Only data that can be accessed are useful, and the depth of water is not one of these. One readily obtainable piece of data is the distance the water surface has receded from its position when the dam is full — a distance obtained simply by stepping out the distance taken to walk from the edge of the dam, directly to the current water level. A key insight is the recognition that this distance (s) is both measurable, and a key input to the model.

### Assumptions

- Water in the dam is reduced by evaporation and in providing for livestock. Seepage loss is negligible and can be ignored.
- Dams whose surfaces are bounded by curved perimeters and with access for animals can be represented by circular or elliptical approximations.
- The dimensions of the dam when empty (as excavated) are known.

### Model for dam volume

The aim is to create a formula that enables the volume to be estimated in terms of the arc length AP(s) which we can measure by stepping to the edge of the water, from the high water mark when the dam is full. (The distance 's' needs to be related to the depth of water 'h' which cannot be measured directly.)

The volume of the cap of a sphere of depth 'h' is given by the formula $V = \pi h^2(3R - h)/3$, where R is the radius of the parent sphere. (This result can be provided after students have identified a need for it, or can otherwise be obtained as an application of integral calculus.)

From Figure 1 we have from triangle OAC:

$R^2 = r^2 + (R - d)^2$ so $R = (r^2 + d^2)/2d$

$r = 15$ and $d = 2$ gives $R = 57.25$.

Then V (full dam) = $\pi d^2(3R - d)/3 = 710\text{m}^3$ (approx.) = 710 000 litres.

Similarly, from triangle OPQ:

$R^2 = x^2 + (R - h)^2$ so that $x^2 = 2Rh - h^2$

This links the radius of the water surface and depth, when the dam is partly filled, that is, for various values of 'h'.

So we know how to calculate V when 'h' is known from $V = \pi h^2 (3R - h)/3$, but we need the volume in terms of a section of arc length 's' that we can actually measure.

Hence we need to express 'h' in terms of 's'.

From Figure 1, we get $h = OM - OQ = R - R\cos\theta$.
Hence $h = R(1 - \cos\theta)$ … (1)
Now $\theta = (\theta + \alpha) - \alpha$ where $\theta + \alpha = \sin^{-1}(r/R) \approx 0.265$ and $\alpha = s/R = s/57.25$.
So $\theta = 0.265 - s/57.25$ … (2)
Hence from (1) and (2) we have
$h = 57.25[1 - \cos(0.265 - s/57.25)]$.
Check: When s = 0, h = 2 (rounded from 1.99)

To substitute this expression into the volume formula to obtain V as a function of 's' is routine but challenging and becomes messy:
$V = \pi R^3(2 + \cos(\sin^{-1}(r/R) - s/R))(1 - \cos(\sin^{-1}(r/R) - s/R))^2$.

It is simpler to proceed in two stages: first calculate values of 'h' from chosen values of 's'.
Then obtain corresponding volumes using $V = \pi h^2 (3R - h)/3$.

www.immchallenge.org.au

| s (m) | h (m) | V ($m^3$) | V/capacity (%) |
|---|---|---|---|
| 0 | 2.00 | 710 | 100.00 |
| 1 | 1.74 | 542 | 76.36 |
| 2 | 1.51 | 406 | 57.14 |
| 3 | 1.29 | 297 | 41.77 |
| 4 | 1.09 | 211 | 29.72 |
| 5 | 0.90 | 145 | 20.46 |
| 6 | 0.73 | 96 | 13.55 |
| 8 | 0.45 | 36 | 5.08 |
| 10 | 0.23 | 9.8 | 1.38 |
| 15 | 0.00 | 0.00 | 0.00 |

*Table 2 Dam volume and depth against distance dam water has receded from high water mark*

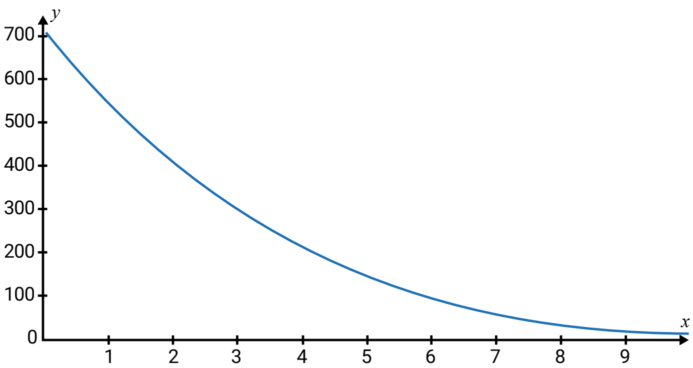



*Figure 2 Graph of volume against 's' for circular dam*

## Solve the mathematics (a) for dam volume

(Note that 's' takes values from 0 to the length of arc (AM) = R($\theta$ +α) ≈ 0.265 x 57.25 or about 15m.)

Example: s = 1 gives h = 1.745 and thence V ≈ 542 ($m^3$). This is approximately 76% of the dam capacity of 710 $m^3$. So when the distance to the water's edge has dropped by only 1 metre (approx 6.67%), the volume has been reduced by almost one-quarter.

Table 2 shows the results of calculating (using a spreadsheet) these quantities for values of 's' at (initially) one-metre intervals. Notice that when the waterline has receded only three metres from the top of the dam, the volume of water is much less than 50% of the capacity.

To interpret the results, Figure 2 contains the graph of $V = \pi R^3(2 + \cos(\sin^{-1}(r/R) - s/R))(1 - \cos(\sin^{-1}(r/R) - s/R))2$ generated by using graphing technology. This could be used, for example, as a chart pinned on the kitchen wall from which the volume of water remaining can be read, given only the distance (s) to the water line from the high water mark when the dam is full.

## Formulate the mathematical model (b) for water loss over time

To estimate how long water in a dam will last, we need to make further assumptions – about evaporation.

**Assumptions**

Internet research will typically show that the evaporation rate varies with temperature, wind speed, sunshine, and relative humidity.

It also varies throughout the year, but a rough daily evaporation rate (average) can be found in (cm or mm) by dividing the average annual value by 365 days. This will be sufficient accuracy for estimation purposes, although in practice there will be seasonal variations.

So estimating the value for central Victoria, from data given in the problem description, we obtain the average amount of evaporation per day = 160/365 ≈ 0.44 (cm).

This is the 'depth' of water that is lost across any exposed surface area in a day. The volume lost will vary with the surface area.

Calculations will overestimate the number of days that suitable water is available to animals. Near the end the dam will resemble a bog and the water undrinkable.

## Solve the mathematics (b) for water loss over time

The daily amount lost by evaporation will vary with the area of water surface, but by using the average value we can estimate how long the water should last without further rain.

From the data provided for this problem, for central Victoria we assume that the average amount of evaporation per day is approximately 160/365 ≈ 0.44(cm). This is the 'depth' of water that is lost across any exposed surface area in a day. The volume lost will vary with the surface area.

From Figure 1, the cross-sectional area when the depth of the dam is 'h' is a circle with radius '$x$' and area $\pi x^2$ where $x^2 = 2Rh - h^2$.

Hence $A(h) = \pi(2Rh - h^2)$.

Note when h = 0, A = 0 and when h = 2,
A = 706.85 – the value of $\pi$ $(15)^2$.

Mean value of the cross-sectional surface area averaged over the interval h = 0 to h = 2 is given by

$$\frac{1}{2-0}\int_0^2 \pi\,(2Rh - h^2)\,dh = \frac{2\pi}{3}(3R - 2) \approx 355.5, \text{ since } R = 57.5.$$

If the dam is full and no more rain falls, we can estimate how many days it would take for the dam to dry up:

710/1.56 ≈ 455 (approximately 65 weeks or 15 months worth of water supply).

If the dam must sustain a herd of 300 sheep on dry pasture, then daily consumption of water by 300 sheep (from the example data) is 2500 × 300/365 ≈ 2055 litres ≈ 2.055$m^3$.

Total average water loss per day from (consumption + evaporation) ≈ 3.62 $m^3$.

Number of days of water available ≈ 710/3.62 ≈ 196 days (about 28 weeks or just over 6 months).

## Interpret the solution

Mathematical outcomes have been continually linked to the dam structure, its volume and dimensions, and practical implications – as for example the meaning of the graph in Figure 2. This is typical of problems involving a variety of mathematical calculations. Their meaning within the problem needs to be interpreted and assessed as they arise, for testing numerical outcomes against the real situation will often identify errors in calculation that need to be addressed.

The formula obtained translates the stepped out distance (s) into a corresponding value for volume that gives estimates of volume for any measured value of s. This would provide the basis for constructing a ready reckoner or wall chart if desired.

The data provided assumed a consistent evaporation rate based on an annual average. But the scenario posed — drought conditions — could be considered to be different from the average. How will the outcome vary, if different values are considered for the evaporation rate?

The data provided for livestock water requirements are annual averages. In a real-world scenario, are animals likely to need more water in drought conditions than in average weather? How will the outcome vary if different values are considered for the water consumption rate?

For water loss over time, noting the observation about boggy conditions when the dam is nearly empty, the figure obtained will overestimate the time drinkable water will be available. A safer estimate would be about one month less. Perhaps?

Students can contribute actively to these sorts of ideas, and resulting refinements, once they engage with properties of the real contextual setting.

## Evaluate the model

Apart from the checking of working for possible errors in mathematical calculations and/or in the application of technology, evaluation involves continuous checking against the needs of the problem context.

Has the solution provided a sufficiently good answer to the problem posed, or do we need further work? Why?

Sometimes when the answer to the first question is 'yes' the first answer obtained suggests a deeper exploration that only becomes obvious from the initial modelling effort. This then stimulates another cycle of modelling with an amended purpose.

A different perspective on evaluation was reported by a teacher who used a version of this problem with her year 10 class. An appreciative parent who happened to be a farmer told her that stepping down the bank was the method he and others used in estimating the amount of water in a dam.

## Report the solution

The modelling report could contain all the above components of the solution of the problem. It should summarise and illustrate how the mathematical insights obtained advanced an understanding of the problem — even if this sometimes means that the solution attempt in its present state is in need of further development. All assumptions and choices of data values should be explained and justified.

In this case the report should develop as a systematic and cohesive narrative: considering the implications of drought conditions for livestock farmers; providing tools for farmers to use to estimate water supply, such as the data shown in Table 2 and the graph in Figure 2; and recommending a timeframe within which provisions should be made for alternative supplies for stock.

# Super Size Me

Disclaimer: This problem is about lifestyle choices. It explores the impact of energy intake (food selection) and exercise on bodyweight, and the particular source of food products is irrelevant.

## Describe the real-world problem

### Australian obesity a big problem

29 May, 2014. Worldwide adult obesity rates have jumped nearly 30 per cent in the last 30 years, according to new analysis from the Global Burden of Disease Study 2013, published in medical journal *The Lancet*. Adult obesity rates in Australia are climbing faster than anywhere else in the world, and are only slightly less than those of the United States. In Australia, 63 per cent of adults are overweight or obese. In response to the study, health experts have called on the governments in Australia to commit to a national strategy to address overweight and obesity.

Commenting on the implications of the study, Professor Klim McPherson from Oxford University in the UK stated that an appropriate global response to the worldwide obesity rates would focus on 'curtailing many aspects of production and marketing for food industries'.

This kind of public concern is not new. In America in 2004, concern over a perceived link between obesity and food industries led to the making of the documentary film *Super Size Me* (2004). For 30 consecutive days, the film's creator Morgan Spurlock Morgan ate three meals daily consisting of nothing but McDonald's food and beverages (consuming approximately 5000 calories daily). If offered an 'upsize' he always took it, and limited his daily exercise to that of the average American office worker. He ate everything on the McDonald's menu at least once. Spurlock, a 32-year-old male who was 188 cm tall and weighed 84.1 kg at the start of the experiment, documented as his weight increased by 11.1 kg. There were other damaging effects to his body.

## Specify the mathematical problem

Develop and evaluate a mathematical model to describe the weight gain experienced by Morgan Spurlock. Use the model to explore the respective effects of calorie intake and exercise. You may find useful information on energy and activity levels in the appendix at the end of this question.

## Formulate the mathematical model

**Data**

- The unit of energy is the kilocalorie (Calorie) where 1 Calorie = 4.18 kilojoules.
- 7700 kcal ~1kg (biological data).
- The average energy intake per day, I, is 5000 Calories (from film description).

### Assumptions

- Energy processing is continuous, with no delays in converting Calories into body functioning, and excess Calories to additional weight. That is, individual daily effects are averaged out over periods of time.
- Weight change (per day) is determined by energy intake (per day) less energy used (per day).
- Energy used per day = the rate at which a body burns energy, that is, the basal metabolic rate (BMR) + energy used in activity (conduct web search to collect data, or see the appendix to this resource).

### Basal metabolic rate

Web search identifies the Mifflin formula, circa 1990, specifying that: BMR (Calories) = 10w + 6.25h – 5a + 5 for males, where w is weight in kilograms, h is height in centimetres and a is age in years.(For females BMR = 10w + 6.25h – 5a – 161.)

Since the experiment lasts 30 days, we assume that age and height can be treated as constants. The film description specifies that a = 32, h = 188, and starting weight = 84.1 kg.
Hence take BMR = 10 × weight + 1020

### Energy used in activity

From the film description we assume a sedentary lifestyle over the course of the experiment, and from the information in the appendix this implies an additional energy usage per day of about 0.2 × BMR.
Hence, energy used per day is 1.2 BMR = 12w + 1224.

### Converting energy to weight

Weight today = weight yesterday + (Intake in calories yesterday – calories used yesterday) converted to weight.

Let $w_n$ be weight after day 'n' (on day 0 original weight is $w_0$ = 84.1 kg).
I = average daily intake of Calories (I = 5000 from film data).
Let $E_n$ be the energy used on day 'n' ($E_n$ = 12w + 1224).

$w_1 = w_0 + (I - E_0)/7700$
$w_1 = w_0 + (3776 - 12w_0)/7700 = 3776/7700 + (1 - 12/7700)w_0 = 0.4904 + 0.9984\, w_0$

Hence $w_1 = a + bw_0$ where a = 0.4904, and b = 0.9984.

Similarly $w_2 = a + bw_1$ and so on… $w_{30} = a + bw_{29}$.

## Solve the mathematics

There are a number of methods that could be used to solve the mathematics.

### Calculator

$w_1$ = 0.4904 + 0.9984 x 84.1 = 84.66
$w_2$ = 0.4904 + 0.9984 x 84.46 = 84.82
and so on.

But this is tedious.

### Spreadsheet

See table below.

Row 3 contains the initial values (day zero).

Formula for BMR is in column E. For example, E4 = (10*B4+6.25*$D$3-5*$C$3 + 5)

Formula for energy used in activity is in column G. For example, G4 = ($F$3*E4)

Formula for weight is in column B. For example, B4 = (H3 – E3 – G3)/7700 + B3

### Geometric series

$w_1 = a + bw_0$

$w_2 = a + bw_1 = a + b(a + bw_0) = a(1 + b) + b^2 w_0$

$w_3 = a + bw_2 = a + b(a + bw_0) = a(1 + b + b^2) + b^3 w_0$ … leading to:

$w_{30} = a(1 + b + b^2 + \ldots + b^{29}) + b^{30}w_0 = a(1 - b^{30})/(1 - b) + b^{30}w_0$

$w_{30}$ = 94.53.

### Calculus

If we treat time as continuous then we can approach a solution through calculus.

Let weight at time t (days) after the start of the 'diet' be w (kg) and consider the change in weight from w to w + δw that occurs between t and t + δt.

If the average daily energy intake is I, then during time interval from t to t + δt intake is ≈ Iδt (Calories).

Similarly, the energy used between t and t + δt ≈ 12w + 1224 (Calories).

Hence δw ≈ [I – (12w +1224)]) δt/7700 so that in the limit dw/dt = (3776 – 12w)/7800.

Thus,

$$\int \frac{dw}{(3776 - 12w)} = \int \frac{dt}{7700}$$

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 1 | day | weight | age | height | BMR | Activity factor | Activity energy | Energy Intake (I) |
| 2 | | (kg) | (yr) | (cm) | (kcal/day) | (BMR multiplier) | (kcal/day) | (kcal/day) |
| 3 | 0 | 84.10 | 32 | 188 | 1861 | 0.2 | 367 | 5000 |
| 4 | 1 | 84.46 | | | 1865 | | 368 | 5000 |
| 5 | 2 | copy | — | — | copy | — | copy | copy |
| 33 | 30 | 94.51 | | | 1965 | | 393 | 5000 |



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We then have $\ln(3776 - 12w) = -0.00156t + c$.

Hence $(3776 - 12w) = Ae^{-0.00156t}$ (where $A = e^c$).

When $t = 0$, $w = 84.1$ so $A = 2767$ and
$w = (3776 - 2767\, e^{-0.00156t})/12$

When $t = 30$, $w = 94.63$

The latter approach would be unlikely to be within the scope of Year 10 students, for whom a spreadsheet solution would be accessible. In general, of course, if a problem can be approached in more than one way then initial insights are confirmed or challenged, and sometimes new insights obtained.

## Interpret the solution

Before leaping to interpretation it pays to reflect back on the context of the modelling. Here a condition was that Morgan consumed approximately 5000 Calories each day. So in reflecting on our solution it would be reasonable to allow the average amount to vary say by 5 per cent from this figure; that is, to carry out calculations assuming average intakes of between 4800 and 5200 calories. What does considering a range of values tell us?

We find that for I = 4800, the predicted weight gain is 9.7 kg (approximately), while for I = 5200 the predicted gain is 11.2 kg (approximately).

Activity level is a more tenuous measure: allowing a 20 per cent tolerance above (0.24) and below (0.16) the value used, with I = 5000 gives a range of weight after 30 days of between 94.2 and 94.8 kg.

It is reasonable to say that the model, built on the basis of daily energy intake (food) and daily energy use (BMR + additional activity) tracks the increase in weight in a logical and coherent way. Given the assumptions it is this property, rather than precise numerical values, that are important for evaluation purposes. (The calculated weight gain for the standard conditions is about 6 per cent lower that the reported figure.)

## Evaluate the model

We should not think of this modelling exercise primarily as an attempt to predict precisely the weight gain of Morgan Spurlock over 30 days. Its purpose, rather, is to develop a model that is applicable in estimating the effects of energy intake and energy use on body weight. The film provides specific data against which to test the model.

Given that it appears sensitive to the 'right' data, and gives results that are consistent with the outcomes reported in the film, we now consider what further insights might be obtained. We can consider some hypothetical circumstances that give insight into the respective impact of food intake and exercise. (This represents a move from the evaluation phase back into modelling activity.)

Suppose we hold energy intake at 5000 and double (increase by 100 per cent) the activity level from 0.2 BMR to 0.4 BMR, assuming that such a diet is compatible with such good intentions. Then we obtain a weight after 30 days of 93.1kg compared with 94. 5. A modest reduction for hard work!

Suppose instead we reduce the daily Calorie intake to 4000 (20 per cent reduction) and the original sedentary lifestyle is maintained. Now after 30 days we obtain a weight value of 90.75 kg, almost 2.5 kg less than the previous value.

Other similar calculations that represent lifestyle decisions drive home the information that diet is the predominant problem, and exercise can only go so far if that is not addressed.

Refinement of the model could be achieved through conducting another scenario.

Sixteen-year-old Sarah weighs a steady 55 kg and had an activity level that is 'lightly active'. She receives a skateboard for her birthday and now skateboards for an hour a day, adjusting her food intake so that her weight remains at 55 kg. Recently some fast food outlets opened near her home, and after skateboarding Sarah now has a daily routine in which she eats a cheeseburger and drinks a 375 ml coke. The rest of her diet remains the same. What is the effect of this habit on Sarah's weight after two months (60 days)?

(Alternatively, replace Sarah's information with a personal choice of weight, food and exercise).

Further information about food content and exercise is provided in the appendix.

## Report the solution

After investigating these problems, write a letter to yourself summarising the understanding you have obtained about the effects of food and physical activity on fitness. Include advice for maintaining a healthy lifestyle.

The letter should summarise briefly the main findings from the two problems, based on Morgan and Sarah. It should highlight insights obtained about the impacts of exercise and diet, and perhaps contain self-directed advice for a healthy lifestyle.

## Appendix

Basal metabolic rate (BMR) is the rate at which energy is used when the body is at complete rest. Use an internet search to find a way of estimating BMR (Mifflin formula).

To find the total energy used it is necessary to estimate the contribution of additional physical activity. There are various ways to do this but a useful one is to express it as a multiple of BMR, which varies with lifestyle.

Examples might include:

- sedentary, little or no exercise, desk job = BMR × 1.2
- lightly active, exercise or sports one to three days per week = BMR × 1.375
- moderately active, exercise or sports six days per week = BMR × 1
- very active, hard exercise every day, or exercising twice per day = BMR × 1.725
- extra active, hard exercise two or more times per day, or training for marathon or other intensive event = BMR × 1.9

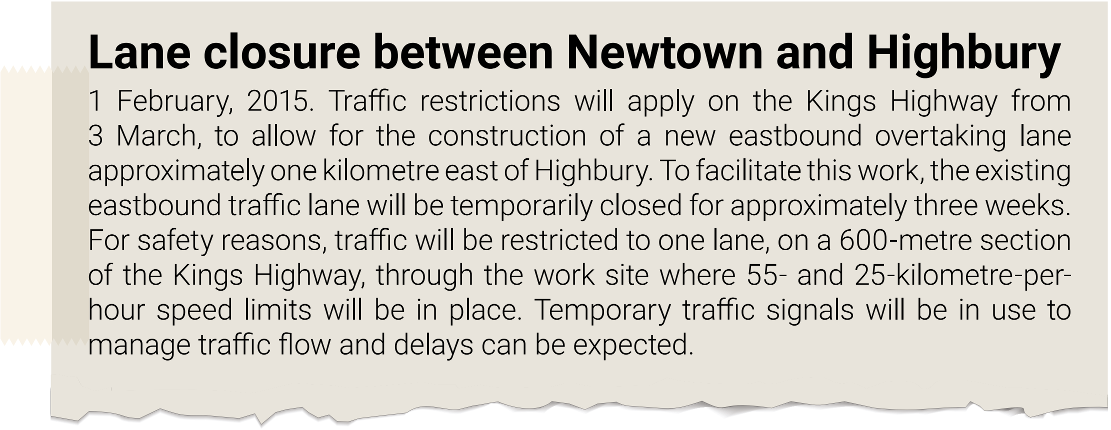
When more energy (calories) are taken in by way of food than are needed by the body, the excess calories are converted to extra kilograms of weight at a rate of 7700 Calories = 1 kg.

| Activity | Energy used (kcal/kg/h) |
|---|---|
| Sitting quietly | 0.4 |
| Writing | 0.4 |
| Standing relaxed | 0.5 |
| Driving a car | 0.9 |
| Vacuuming | 2.7 |
| Walking rapidly | 3.4 |
| Skateboarding | 5.0 |
| Running | 7.0 |
| Tennis | 7.0 |
| Swimming | 7.9 |

| Sample fast food items | Energy (kcal) |
|---|---|
| Plain hamburger: ground beef patty, grilled (40 g), hamburger bun (65 g) and salad | 328 |
| Plain hamburger and medium chips: ground beef patty, grilled (40 g), hamburger bun (65 g) and salad | 497 |
| Cheeseburger: ground beef patty, grilled (40 g), hamburger bun (65 g), lettuce, sliced tomato and onion, tomato sauce, cheese (16 g) | 391 |
| Cheeseburger and large chips: ground beef patty, grilled (40 g), hamburger bun (65 g) and salad, large portion of chips | 728 |
| Double hamburger and medium chips: 2 ground beef patties, grilled (80 g), hamburger bun (65 g) and salad, medium portion chips | 606 |
| Pizza with cheese, tomato and olives: medium portion (90 g) | 223 |
| Pizza with cheese, tomato and olives: large portion (340 g) | 844 |
| Lemonade (375 ml) | 158 |
| Dry ginger (375 ml) | 124 |
| Bundaberg ginger beer (375 ml) | 188 |
| Coca Cola (375 ml) | 158 |
| Pepsi cola (375 ml) | 169 |
| Fanta (375 ml) | 210 |

# Temporary traffic lights

## Describe the real-world problem



### Lane closure between Newtown and Highbury

1 February, 2015. Traffic restrictions will apply on the Kings Highway from 3 March, to allow for the construction of a new eastbound overtaking lane approximately one kilometre east of Highbury. To facilitate this work, the existing eastbound traffic lane will be temporarily closed for approximately three weeks. For safety reasons, traffic will be restricted to one lane, on a 600-metre section of the Kings Highway, through the work site where 55- and 25-kilometre-per-hour speed limits will be in place. Temporary traffic signals will be in use to manage traffic flow and delays can be expected.

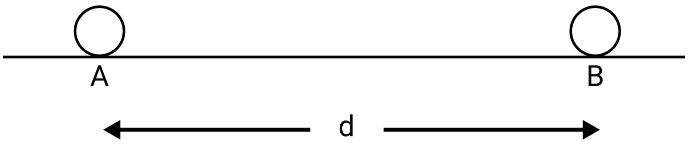
## Specify the mathematical problem

Two-way traffic along a 600m single lane section of roadway is controlled by temporary traffic lights at each end. A speed limit of 55 km/h applies. How should the timing of the lights be set to achieve an efficient flow of traffic in both directions?

## Formulate the mathematical model

We first need to understand how the light cycle operates.

- Traffic lights are at A and B, where d = 600, and V (max speed) = 55km/h.
- Assume that the road is a main highway – no pushbikes or farm machinery allowed.
- Assume that the traffic is similar in both directions and sufficiently dense to build up while the light is red.
- Last car through on a green light at A will take (d/V + safety margin(s)) sec to clear the road, before the light at B turns green
- Let T (same for both) be the green light time at A and B.




| Complete cycle of light changes at A (same for B) | | | | | |
|---|---|---|---|---|---|
| Time | 0 | T | T + d/V + s | 2T + d/V + s | 2T + 2d/V +2s |
| Light at A | G | R | R | R | G |
| Light at B | R | R | G | R | R |

## Solve the mathematics

The length of cycle is $2(T + d/V + s)$
so, the proportion of green light time = $T/2(T + d/V + 2s)$.

Using V = 15 m/s (54km/r) and and a safety margin of 5s gives proportion of green light time = $T/2(T+45)$.

For example: T = 60 gives a value of 0.29 (29%)
T= 180 gives a value of 0.4 (40%)

## Interpret the solution

Does this mean that longer green light times (at both ends) are more efficient?
We don't know.

What are the implications for traffic flow?
Our model so far is not wrong, but we evaluate it as inadequate to answer this question, as yet we have no basis for assigning values to T.

How do we obtain estimates of the respective numbers of vehicles arriving at the lights, and leaving while they are green?

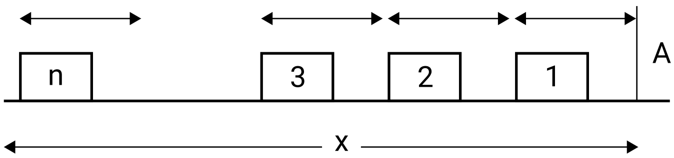
First, consider vehicles arriving at A during a cycle of light changes.

Defensive driving advice says to leave 2s between vehicles passing a roadside marker.

On a busy approach road this gives an estimate of the rate at which vehicles arrive, and typically join a waiting queue. Keeping things general, we denote this parameter by b.

So, the number of vehicles arriving during a cycle of length $2(T + d/V + s)$ is given by
$N = 2(T + d/V + s)/b$
that is $N = 2(T + 45)/b$

A model for vehicles leaving A during green light time is shown below.




Next, consider vehicles waiting for green light.

Average vehicle length (including space to vehicle in front) = l.
Vehicle n is last through the light when it turns green.
a is the average delay time between successive vehicles taking off.
So available time in motion for nth vehicle to reach lights = $T - an$.

Assume that we can ignore short period of acceleration to speed V.
Then $ln = V(T - an)$ and so $n = VT/(l + aV) = 15T/(6 + 15a)$.

For cars to just clear during a cycle of light change we need n = N.

Thus $15T/(6 + 15a) = 2(T + 45)/b$
So $T = 90(2 + 5a)/(5b - 10a - 4)$

We have noted above a reasonable (minimum) estimate for b: $b \approx 2$.

Drivers are 'rearing to go' when vehicle in front moves so estimate $a \approx 0.5$.

This gives $T \approx 405$.

## Evaluate the model

Nobody is going to sit comfortably for 6 to 7 minutes in a queue with no sign of activity; so, review the outcome.

Do we need a new model or to amend some aspects of the approach?

Suppose T = 240 (4 min): n = 267 and N = 285 (18 vehicles wait a turn)
T = 180 (3 min): n = 200 and N = 225 (25 vehicles wait a turn)
T = 60 (1 min): n = 67 and N = 105 (38 vehicles wait a turn)

But now the percentage missing out is approaching 40 per cent, and the build-up will be rapid. So balance driver frustration with volume of traffic movement. How many light changes are tolerable on a busy road: Two? Three? Four?

Conduct a retrospective estimate of omitting acceleration phase: If T = 180, the nth vehicle (number 200) will have travelled 200 × 6 = 1200 metres when it just makes the light, and will have been travelling for about 80 s. The initial phase is a very small fraction of this.

Revisit parameter values: l, s, a, b.

Note from the formula for T that we must have: $5b - 10a > 4$.

Consider different times of day: the effect of b.

Consider the influence of heavy truck presence.

Investigate the effect of including an acceleration phase, for example:
T = 240: n = 263 versus 267 (change 1.4%)
T = 180: n = 197 versus 200 (change 1.5%)
T = 60: n = 63 versus 67 (change 5%)

## Report the solution

The modelling report should contain all the above components of the modelling problem and its solution. The report should synthesise this data into a cohesive narrative, considering the implications for safety and driver frustration. The report could recommend a particular option that best balances the factors of wait time and traffic build up.

# Collaboration and communication

Substantial mathematical modelling projects (such as IM²C projects) are best done in groups. However individual group members need to be competent in all stages of the modelling process, with expertise developed through prior experience with a range of modelling problems. In terms of the IM²C, we suggest that teams of four are valuable in view of the number of tasks that have to be covered in producing answers and a report. However, this and later advice, is relevant to consider for all modelling contexts, whether IM²C related or not.

It can be worthwhile assigning different members of a team to different tasks – again this depends on the expertise of members. There are the 'mathematicians' who go from the assumptions to the mathematics; the 'project managers' who look after and keep track of overall progress and make sure that a summary or report is written, that the assumptions are reasonable and relevant, that the mathematics is correct, and that the conclusions follow from the mathematics and answer the posed problem. Different members of a team may do more than one task. But it is also important that role assignment remains flexible, as often members of a team can contribute at times to areas that may not be their prime responsibility, and this needs to be provided for.

Report writing is fundamental to modelling. For teams in the early stages of modelling, the report may be as simple as a series of bullet points that lead to the problem solution. As teams mature, they should be encouraged to write a full summary that will state the problem, indicate all assumptions, give some idea of the maths used and provide a conclusion. Gradually students should work up to a full report with details of all the mathematics and all the sources that were used to complete the problem.

To help students in their report writing, we suggest that regular classwork should include pertinent explanations as well as symbols and equations. It will be easier to see how students reason and their general understanding and fluency will be enhanced.

### Selecting a team

Successful teamwork in mathematical modelling depends both on team composition and on the quality of collaboration between team members.

Modelling is a cyclic process involving different tasks in different stages (see the mathematical modelling framework, p. 6–7).

It is important that all team members have knowledge that enables them to contribute (perhaps in different ways) to the substance of the mathematical modelling involved.

For example, if one or two individuals provide and argue for a particular mathematical approach, others need to supply critique, both of the correctness of the mathematics and of its relevance to the aspects of the modelling task for which it has been proposed.

Teams of students can collaborate by bringing complementary perspectives to paired problem-solving (one an initiator and the other a checker).

The evaluation stage in the modelling process is crucial: strong opinion, if not underpinned or resisted appropriately with mathematical argument, can derail the best of intentions.

A team will have more avenues to pursue if its members know more mathematics – but advanced mathematics alone will not produce a successful report. The team should also include members who know what is required of the modelling process, and who realise the importance of mathematics in industry, science, and life generally.

### Using technology

It will be helpful for some members of a team to have an awareness and facility with technology. The ability to write computer programs is also valuable.

Technological tools do not simply amplify cognitive processes – they can fundamentally change the nature of a task and the requirements to complete it.

Graphics calculators or software such as Matlab, Maple and Mathematica lead to less tedious and more efficient execution of calculus and other types of problems, and generate tables, graphs and so on.

Mathematical expertise should not be considered simply as the accumulation of internal mental processes and structures, but also as a process of appropriating tools that transform tasks, and the relationship of individuals to them.

The most powerful use of technology for problem-solving is in enabling individuals, including students, to explore ideas and tackle solution paths that would otherwise be beyond them.

For the IM²C, each team should plan to have access to at least two computers loaded with software such as a computer algebra system and spreadsheet program, and team members who are proficient in these tools.

### Collaborative competence

It is possible to consider teamwork from a Vygotskian notion of a 'zone of proximal development' (ZPD).

The zone of proximal development is the gap between what a student can achieve working alone and when aided.

The theory posits that education is most effective when it is neither too easy nor too hard; when tasks challenge an individual to stretch past existing competence while support or scaffolding the learner to advance.

The Vygotskian notion of a zone of proximal development (ZPD), through which individual learning is scaffolded by utilising differential levels of expertise existing between expert and novice participants, has been extended to apply within collaborative group activity (see e.g., Goos, Galbraith and Renshaw).[10] Unlike the traditional scaffolding notion, this view of the ZPD involves egalitarian relationships. It refers to activity in peer groups where each member brings some relevant knowledge and/or skills, but requires others' contributions to build on or refine these attributes for effective group progress to be made. Such situations approximate practices within mathematical teams that are striving to go beyond the established boundaries of their knowledge, and so provide authentic experiences of doing mathematics under conditions of uncertainty, as modelling indeed sets out to do.

Importantly within such teams, when the direct influence of the teacher is removed, students must take personal responsibility for the ideas that they are constructing and testing, so that authorship of mathematical knowledge is then vested in individuals and their partners.

If recognition of peer contribution to a common task doesn't happen, the team cannot function properly. It therefore follows that social factors involving relationships and respect between group members are also factors in constructing effective teams.

## Timelines and tasks

In preparing for tasks such as those provided by the IM$^2$C, and modelling projects in general, a routine should be rehearsed in practice situations, using models obtained from other sources well before the challenge problem itself is undertaken.

Team members should understand the routine so that it can be put into action automatically over the days of the challenge.

### Timelines

- Plan an overall approach and timeline. Make it flexible within limits.
- Take note of the total time available for the project.
- Schedule meeting times to share and review progress. It may not be feasible to schedule too far in advance: students typically schedule meetings at unusual times outside school hours to suit individual availability.
- Team members should work together for at least the first and final days of the challenge period, because of the importance of the initial planning and the final report writing.
- Agree on a time by which all raw modelling activity needs to be completed.
- Times are nearly always underestimated, so set an early deadline for the first tasks.
- Plan sufficient time for report writing, which is likely to take at least a day.
- Meet or video conference once every day to discuss progress and next steps.

### Documentation

- Maintain a record of all sources, websites and pieces of mathematics. Be careful and systematic, and update continually, so information can be easily retrieved if required in the final report.
- Back up documentation and drafts frequently.
- Don't throw anything away, even apparently redundant copies of electronic files – you never know when something might come in handy. It is always possible that your latest good idea may not be as good as something that you might have discarded earlier.

### Order of tasks

- To begin, the team should come to a mutual understanding of meaning of the first mathematical question to be addressed.
- Decide and allocate initial internet searches to be undertaken, but allow time for additional searches as you get deeper into the problem and the report.
- Consider relevant technology. Search online for specific programs, and for data to support that provided in the problem statement.
- Assign sub-tasks that can be carried out in parallel by different team members, and ensure all individual work is brought together for critique and integration.
- Midway through the challenge period, the team should begin to collate members' work, draft a summary and outline the maths used.
- Begin to consider the advice the team will give in response to the challenge question being posed.
- Everyone in a team should be responsible for critically assessing what is written as it is written.
- Errors and approaches need to be discussed. Were the assumptions that were made useful or too strong? Were all the assumptions used? Were there implicit assumptions that should be stated explicitly? Could the problem be approached in a different way?
  Are more data needed?
- Collect all references together as a list and think about the need for one or more appendices.

[10] Goos, M., Galbraith, P., & Renshaw, P. (2002). Socially mediated metacognition: Creating collaborative zones of proximal development in small group problem solving. *Education Studies in Mathematics*, 49(2), 193–223.

- Identify what work still needs to be done to answer the challenge problem.
- At least one full day should be devoted to report writing.

### Teacher/mentor assistance

The role of the teacher or mentor is to guide students to become proficient modellers.

The teacher should scaffold of task implementation, without providing specific advice on the problem.

Teachers might direct students to the framework for modelling, or give suggestions on time management.

For formative modelling tasks outside the $IM^2C$, a teacher can exercise discretion in deciding how direct hints should be.

The aim is to help students overcome blockages to progress, but not compromise critical decision-making and learning processes.

### Writing a modelling report

Models are designed to address particular problems in specific situations. So, models must be capable of being evaluated and used by others, including non-mathematicians.

Information needs to be communicated clearly and fully.

While they should always contain information that gives a complete picture of what the modelling has achieved – and not achieved!

The $IM^2C$ requires that a team report be produced in which the contributions of individual members are merged. For modelling undertaken within other contexts (e.g. school based activity) other reporting styles can be employed. For example, when assessment of individual expertise is required, an effective approach has been for students to work collaboratively, but be responsible for providing an individual report. The checklist below is relevant to any modelling project.

While reports on mathematical models vary in style and detail, they should always contain information that gives a complete picture of what the modelling has achieved – and not achieved!

#### Report writing checklist

A checklist for a good report may include:

- Describing the real-world problem being addressed.
- Specifying the resulting mathematical questions precisely.
- Listing all assumptions and their justification.
- Indicating sources of imported information (for example, websites).
- Explaining how numerical values used in calculations were decided on.
- Showing and justifying all mathematical working.
- Setting out all mathematical working, graphs, tables, etc.
- Interpreting mathematical results in terms of the real-world problem.
- Evaluating the result – does your answer make sense? Does it help to answer the problem?
- Dealing with refinements to the original problem.
- Qualifying the solution.
- Recommending the solutions arising from the work. What further work is needed?

In practice, several of these activities can occur at the same time. For example, obtaining a mathematical result, interpreting it, and evaluating its correctness or relevance, are aspects that are often dealt with together.

Evaluating the solution(s) in terms of the problem requirements is crucial. It is the final crescendo in this orchestral work. It is here that reporting to the client is to be found. In the heart of a report a team may use all the complicated mathematics and modelling that they know or that they can find out, but it is necessary to communicate the results to the client. If this is not done clearly the whole point of the exercise is lost.

#### Report summary checklist

Writing reports is not easy. It takes a great deal of practice to produce reports of value. Students should be guided to practice writing reports before the challenge.

The mathematics and the modelling should not, in themselves, be the focus of the report. The focus of the report should be a solution to the real-world problem, and the mathematics and the modelling are the means to this end.

The 2016 IM2C required participants to submit a one-page summary sheet and a solution of up to 20 pages. A well-written summary is vital. **Consider the summary as a direct statement to the people who needed to solve the real-world problem**. Be aware that the end users' mathematical ability may not be high, so the practical suggestions may need to be in everyday language.

The summary should:

- state the problem
- state the assumptions made
- give a brief description of the mathematics used
- provide practical suggestions to solve the real-world problem.

The summary might also discuss what might have been done to develop the solution further on another occasion.

# Modelling report illustrating use of criteria

The example in this section illustrates a modelling report from an Australian competition (not related to the International Mathematics Modeling Challenge (IM²C)). The purpose of this example is to illustrate the general applicability of the criteria used for the Australian component of the IM²C.

The AB Paterson Gold Coast Modelling Challenge was devised and hosted annually by Trevor Redmond for a period of approximately 10 years. Its senior section involved Year 10 and Year 11 students from secondary schools, almost all new to modelling, working in mixed-school teams during a two-day modelling challenge. Student teams (most commonly comprising four members) were assigned from different participating schools. The challenge commenced with an introduction to modelling by mentors who worked with five or six teams in a class setting for approximately two hours on a common modelling task chosen by the mentor. The purpose here was to provide a common understanding of what is involved in modelling and reporting. The introduction class covered the cyclic modelling process by applying it to the problem chosen for this purpose. Following this introduction, each student team chose their own real situation to investigate, devised a mathematical problem to address, and worked collaboratively until lunchtime on the second day to complete their modelling and construct a poster or report describing their work. Each team gave an oral presentation within the class group in the post-lunch session, followed by a public display of posters from all groups later in the afternoon.

Mentors facilitated students' pathways in modelling, rather than specifying approaches or choosing situations or questions for groups to model. The mentors intervened as little as possible. Their role was to enable students to develop and evaluate their own ideas in a productive manner. For example, mentors would answer questions asked of them by students, draw attention to where teams were in relation to their modelling progress, and what this might mean going forward. They did not provide specific advice as to how to proceed mathematically towards a solution or direct students down a particular path.

The Gold Coast Challenge has both similarities to and differences from the IM²C. The Gold Coast Challenge was similar to the IM²C in the enactment of a cyclic modelling process (e.g., making and justifying assumptions; choosing the mathematical approach; interpreting mathematical outcomes in terms of the real-world context; evaluating progress in terms of problem requirements; compiling a report and justifying results or recommendations). It differed in that the students selected their own problem context rather than having it prescribed. Students also had substantially less time (1.5 days plus overnight if students chose to use that opportunity) for completion and report construction. This meant, for example, that only one mathematically based question could be generally addressed, and that evaluation of outcomes could be foreshadowed and discussed but not always pursued, and the outcomes could not always be fed into a second round of modelling.

What follows is an example from the Gold Coast Modelling Challenge, annotated with mentor comments. This single example illustrates, in a targeted way, the attributes that are important in developing and reporting on any modelling problem. Thus, it contains features that are consistent with, but less extensive than, those that are expected within a complex context such as the IM²C. The performance of the team is discussed in terms of the criteria used for judging within the 2016 Australian component of the IM²C, which were not known to student teams involved in the IM²C, nor of course the students here.

This example can be viewed as providing a bridge to the more extensive demands of the international challenge. Alternatively, for those not intending to go the full journey to the latter, it provides a self-contained illustration essential components of the development and reporting of a modelling problem.

The material that follows is illustrative. In using selected excerpts from student work, it does not set out to demonstrate a perfect solution. The focus is on how the students systematically introduced and addressed essential phases of the modelling process for a problem selected by them. It thereby demonstrates that the Australian criteria developed for grading modelling projects are workable.

In the excerpts that follow 'student' is used to identify material included in the team report, while 'mentor comment' indicates feedback from a competition mentor or the authors. Note, as indicated above, that the headings and associated text outline the major modelling steps.

*Australian judging criteria for the International Mathematics Modeling Challenge 2016*

| Criteria | Sub-criteria |
|---|---|
| **1** Problem definition | ■ specification of precise mathematical questions from the general problem statement |
| **2** Model formulation | ■ identification of assumptions with justification<br>■ choice of variables<br>■ identification and gathering of relevant (needed) data<br>■ choice and justification of parameter values<br>■ development of mathematical representations |
| **3** Mathematical processing | ■ application of relevant mathematics<br>■ invocation and use of appropriate technology<br>■ checking of mathematical outcomes for procedural accuracy<br>■ interpretation of outcomes in terms of the problem situation |
| **4** Model evaluation | ■ adequacy and relevance of findings in relation to problem situation<br>■ further elaboration or refinement of problem<br>■ relevance of revised solution(s) following revisiting and further work within earlier criteria<br>■ quality of answers to specific questions posed in problem statement |
| **5** Report quality | ■ summary page quality: succinctness, power to attract reader<br>■ overall organisation of the report: logical presentation including<br>● description of the real-world problem being addressed<br>● specification of the mathematical questions<br>● listing of all assumptions wherever they are made<br>● indication of how numerical parameter values used in calculations were decided on<br>● setting out of all mathematical working: graphs, tables; technology output and so on<br>● interpreting the meaning of mathematical results in terms of the real world problem<br>● evaluating the solution(s) in terms of the problem requirements |

# Gold Coast inundated

## Problem definition

### Students: Rationale for choice of modelling topic

Climate change is the term used to describe the changing nature of the world's weather patterns. Many meteorologists have stated that climate change could potentially result in detrimental impacts upon our current way of life, through rising global temperatures and sea levels. The increasing temperatures due to climate change have been reported to cause rising sea levels due to the expansion of water around the world due to heat (including the melting of ice). The concept that the climate is changing is strongly supported by evidence obtained by experts.

### Students: Mathematical question

At what point in time will the Q1 (or a later building) lobby, which is 3m above sea level, be submerged due to sea level rises, and what will be the mean maximum temperature of that month in Surfers Paradise at that time?

(Note that this is slightly different in nature to the IM²C in that this problem, with the above wording, was determined by the students themselves.)

### *Mentor comment*

The students have identified an issue they deem significant, and the mathematical question is personally relevant given they live on the Gold Coast and are familiar with the Q1 building. The question is very specific given the tools available, as becomes apparent as the solution attempt proceeds, and revisiting becomes necessary as part of the cyclic modelling process.

## Model formulation

### Students: Assumptions

It is assumed that the influence of rainfall and evaporation in the ocean is negligible due to the fact that this would only contribute to the water cycle which would in turn feed back into the oceans. If this is not the case, the process used to determine the increase in sea level from the temperature increase would not be able to be used to produce accurate results.

It is assumed that the whole surface of Surfers Paradise is a flat plane 3 metres above sea level that contains no obstructions to the path of the ocean as it rises. If this was not the case, the calculations for the rise in sea level would be too simplistic to be valid, and there could be a significant difference between them and the actual sea level increase necessary to reach the lobby of the Q1.

It is assumed that the trend demonstrated in the data set used continues into the future. If the trend does not continue into the future, it would not be possible to accurately predict data points.

It is assumed that there is a correlation between temperature and rising sea levels; if there is no correlation, it would be illogical to attempt to predict the change in sea level using the temperature of Surfers Paradise.

It is assumed that the melting of the polar ice caps does not contribute to the rise of the sea level and that rising sea levels are entirely caused by expansion of water due to heat. If this assumption proves invalid, the predicted point in time at which the Q1 lobby would be at sea level would be too far in the future, as melting polar ice could increase the rate at which the sea level rises; making the predicted time frame for the Q1 lobby to be at sea level inaccurate.

### Students: Defining the variables

The independent variable, $\chi$, is the time in months, in one month intervals, since January 1938. Therefore: $\chi = 0$ = January 1938; the first month that temperature records are available.

The dependent variable, $\gamma$, is the mean maximum temperature of Surfers Paradise over one month measured in degrees Celsius.

### Students: Building a model

The general form for a periodic function is $\gamma = a\sin 2\pi/b(\chi - c) + d$, where a is the amplitude, b is the period, c is the phase shift, and d is the vertical translation of the function. Thus it would not be possible to predict a change in the mean maximum temperature of Surfers Paradise unless an equation representing the general change in the climate was used in place of the constant d value.

To obtain the equation for d, one must determine the average rate at which the temperature increases. A linear regression performed upon the data gave an equation of $\gamma = 0.001\chi + 24.64$. This equation shows that the equilibrium line of the periodic function for the mean maximum temperature over time graph is sloping upwards.

This linear equation can be substituted into the general periodic equation instead of d.

The a value in a periodic equation is the amplitude of the wave; essentially half the range of values for the $\gamma$ axis.

The b value in a periodic function is the period of the function being the average distance between two crests (or troughs) on a wave. Calculating the period as 12, the b value for the periodic equation is $2\pi \div b = 0.5235$.

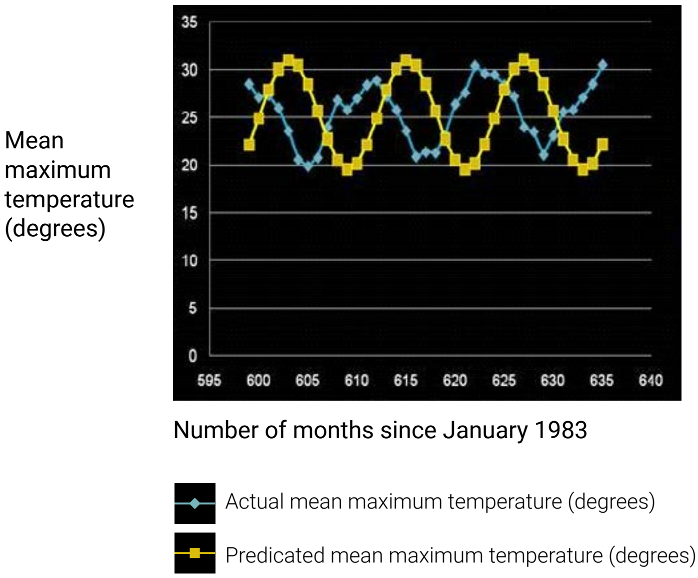
To determine the value of c, the equation can temporarily exclude the c value and be used to predict values for $\gamma$. Because these values will have been translated incorrectly due to the lack of a c value, they can be matched up with corresponding $\gamma$ values and the $\chi$ distance between these selected corresponding points can be used to determine the value of c.

One pair of crests and one pair of troughs have been identified for analysis. The adjacent troughs have $\chi$-values of 605 and 609, respectively – indicative of a horizontal phase shift of four months. This indicates that predicted $\gamma$ values have been shifted four months forward, and thus the c value is –4.

Thus the equation to predict the mean maximum temperature over a month in Surfers Paradise becomes

$$\gamma \approx 5.75 \sin(0.5235(\chi+4)) + (0.001\chi + 24.64).$$

*Actual mean maximum temperature (degrees) and predicted mean maximum temperature (degrees) without a c value over the number of months since January 1938*



*Mentor comment*

The assumptions are careful and relevant, foreshadowing potential limitations as well as introducing necessary simplifications to enable mathematical formulation.

Relevant data were identified as those available from official records kept from 1938.

The independent and dependent variables are appropriate – other subsidiary variables and parameters were introduced as the modelling progressed. An annual periodic relationship was assumed between temperature and time of year based on real world seasonal cycles.

Arguments supporting choice of parameters for the periodic function were presented carefully. Some innovative thinking is apparent – in particular, the replacement of 'd' by a linear expression to capture the slowly increasing base temperature.

There is some loose description that entangles the period with the coefficient, but the students have the mathematics right.

Their incorporation of the translation constant 'c' is clever, even though subject to error. They inferred a value of 4 empirically from the graph of data – rather than a value of 3 (1/4 of a period). The numerical impact on outcomes is minimal.

The final criterion in this set (development of mathematical representations) merges with the first criterion of the following group below (application of relevant mathematics). There is interplay between these activities throughout the remainder of the solution process, and it is natural that this should occur. The important aspect is that both are identifiably present.

### Students: Verification of the periodic model

This perception of the model being fairly accurate (e.g., graph), can be verified by independently comparing a single predicted data point with a single actual data point. Below, the model is used to generate the mean maximum temperature of Surfers Paradise over the 80th month from January 1938.

$y \approx 5.75 \sin(0.5235(x + 4)) + (0.001x + 24.67)$

$y \approx 5.75 \sin(0.5235(80 + 4)) + (0.001 \times 80 + 24.67)$

$y \approx 5.75 \sin(0.5235 \times 84) + (0.08 + 24.67)$

$y \approx 5.75 \sin(43.974) + 24.75$

$y \approx (5.75 \times 0.6943) + 24.75$

$y \approx 3.992 + 24.75$

$y \approx 28.74$

This calculation, however, produces a value of 24.702291 when performed automatically by a calculator due to the fact that it has not compounded errors due to rounding. The actual value of the mean maximum temperature in the 80th month from January 1938 was 24.8 degrees. Thus, the calculated value was off by approximately 0.0977 degrees, while the value calculated by hand with rounding errors using the model was off by 3.94 degrees. The fact that the model, when used with a computer, produced a result that was incorrect by less than one tenth of a degree in this particular case would indicate that it is quite accurate. Based upon this, it would be reasonable to assume that the model could accurately predict the future trend of the mean maximum temperature of Surfers Paradise.

*Mentor comment*

The students first sought to test the accuracy of the periodic function constructed to predict temperatures into the future. An application of mathematics which preceded a direct attack on the problem question itself. This amounted to checking the quality of a tool before using it for its designed purpose – a wise thing to do. The students identified an error in a hand calculation that was ascribed to a rounding error – in fact a mentor check showed that it was because the calculator mode was set to degrees instead of radians! However, the team appropriately discarded it in favour of a computer-generated calculation. The students used calculators and computers appropriately for generating functions, displaying them graphically, and undertaking messy computations that would have been impossible using real data without technology to assist.

The students recognised that output from a model of this type should be checked against real data, but missed that the checking should be over a cycle rather than for a single point. If the point happens to be where the cosine/sine has value 0, then that term will not be evaluated – this happened here.

## Mathematical processing

### Students: Solving the mathematical problem

According to Baltimore City College (internet source provided by students) the volume of water will increase by 0.0088% when heated 10 degrees. The Q1 lobby is only approximately 3 metres above sea level, and it is unlikely that the world temperature will rise by 10 degrees due to global warming, so this value must be made smaller. To determine the expansion of water at temperatures lower than 10 degrees, a ratio must be used as seen on the poster. Therefore, for each degree of temperature increase, the world's water will expand by 0.00088%. The volume of the Earth's water is approximately 1.3 billion cubic kilometres, and its surface area is 361 million square kilometres (reference given by students). Because the volume of a rectangular prism is known, and the world's oceans have already been put in terms of a rectangular prism, it is possible to calculate the increase in the height of the ocean by using the formula below (see continuation of solution).

*Mentor comment*

The Baltimore reference is one of many that can be used for this purpose. The students did not list in among references which was an omission on their part. In addressing the modelling question, the team developed further assumptions, for example collecting the oceans into a cuboid to make the calculation of estimates tractable, and researching internet sources to provide key parameter information on water expansion to feed into the calculation. A clever first choice but a crude one. It enabled an order of magnitude estimate. It was suggested in later discussion with the team that there were other ways that might be used to approximate local water volumes to the same purpose more appropriately. They saw the point but there was not time to pursue the implications.

**Students: Solving the mathematical problem (continued)**

$v = lwh$
(temperature increase × 0.00088) × (1 300 000 000) = 361 000 000 000$h$
(temperature increase × 0.00088) × (1 300 000 000) ÷ 361 000 000 000 = $h$
0.003169 × temperature increase = $h$

*Mentor comment*

Here the students have collapsed some calculations and scaling in the one procedure. They have obtained a correct outcome but without adequate explanation. Their approach was as follows:

expansion volume = temperature increase (degrees) × (expansion/degree) × present volume

expansion volume = area of water surface × expansion height (h)

So (temperature increase) × 0.00088 × 1 300 000 000 = 361 000 000 × h (km)
Leads to h = 0.003169 × temperature increase (metres)

The mathematics is correct, but its communication leaves something to be desired.

**Students: Solving the mathematical problem (continued)**

Therefore, one can calculate the increase in sea level when given the increase in temperature in degrees Celsius. Assuming a height increase of at least 3 metres to reach the lobby of the Q1, the equation becomes:

0.003169 × temperature increase = 0.003
temperature increase = 0.003 ÷ 0.003169
temperature increase = 0.946671

Therefore, the temperature increase, necessary to create a rise in sea level of 0.003 kilometres, or 3 metres, is 0.946671 degrees Celsius. Of course, for this change to actually take effect the change in temperature must be substantial and sustained; it should be the mean of the mean maximum temperatures generated by the model. Due to the periodic nature of the model, the equation should have a linear regression performed upon it to determine an equation for its equilibrium line, thereby allowing one to determine the point in time at which the mean of the mean maximum temperatures will have increased to the point at which the sea level will have risen by 3 metres.

A linear regression performed upon the results of the mathematical model produced an equation of $y = 0.001\chi + 24.64$, where y is the mean of the mean maximum temperatures, and $\chi$ is the number of months since January 1938. Using this linear equation, the number of months since January 1938 at which the sea level would rise by 3 metres can be calculated.

The current number of months since January 1938 is

((2009 × 12) + 11) − (1938 × 12) = 24119 − 23256 = 863 months, and the most recent mean temperature is 25.275 degrees.

With a temperature increase of 0.946671 the temperature would need to be
25.275 + 0.946671 = 26.2217 degrees. Substituting that value into the equation the formula becomes
$26.2217 = 0.001\chi + 24.64$.

Below, the equation is algebraically rearranged to solve for $\chi$.

$26.2217 = 0.001\chi + 24.64$
$26.2217 - 24.64 = 0.001\chi$
$1.5817/0.001 = \chi$
$1581.7 = \chi$

Rounding $\chi$ up to 1582 months since January 1938, the number of years since January 1938 is 131.833, or approximately 132 years. Adding that on to 1938, the result is that in the year 1938 + 132 = 2070, the Q1 lobby will be at sea level, with the mean of the mean maximum temperatures of that year at 26.2217 degrees, and the mean maximum temperature of that month at
$y = 5.75 \sin (0.5235(1587.1+ 4)) + ((0.001 \times 1581.7) + 24.67) =$ 24.19 degrees.
This will occur in 2070 − 2009 = 61 years' time.

*Mentor comment*

As noted above, in addressing the modelling question, the team developed further assumptions, for example collecting the oceans into a cuboid to make the calculation of estimates tractable, and researching internet sources to provide key parameter information on water expansion to feed into the calculation.

Using the formula previously generated, predictions were calculated and interpreted. The interpretation of the outcome triggers the next part of the analysis.

## Model evaluation

**Students: Review of solution**

After reading over the process previously applied to determine the amount of time taken for the lobby of the Q 1 to be at sea level, it became apparent that the equation to calculate the sea level increase when given the temperature increase was incorrect, as the percentage of water expansion was not correctly placed inside the equation; it should have been decreased by a factor of 10 to allow one to multiply the percentage by the temperature increase. This is taken into account in revised calculations: temperature increase = 9.46671 leading to 10101.7 = $\chi$.

Rounding $\chi$ up to 10102 months since January 1938, the number of years since January 1938 is 10102 ÷ 12 = 841.833, or approximately 842 years. Adding that on to 1938, the result is that in the year 1938 + 842 = 2780, the Q1 lobby will be at sea level, with the mean of the mean maximum temperatures of that year at 34.7414 degrees, and the predicted mean maximum temperature of that month will be

$\gamma = 5.75 \sin (0.5235(10102 + 4)) + ((0.001 \times 10102) + 24.67) =$ 35.053 degrees.
This will occur in 2780 − 2009 = 771 years' time.

The model predicts that there will be a sea level rise of 3 metres over 771 years. According to meteorologist Kurt Wayne, the sea level will rise 'by 15cm by 2030'. Assuming a constant rate of sea level increase, this could be interpreted as

15cm ÷ 10 years = 0.75cm/year
300cm ÷ 0.75cm/year = 400 years

So Kurt Wayne predicts that in 400 years, the sea level would rise by 3 metres, assuming a constant rate of sea level rise. The model created does not however, take into account the melting of polar ice caps; something that could contribute significantly to the rising of the sea level. The difference of 311 years between Kurt Wayne's prediction and the model created could be reasonably suggested to be due to the fact that the model generated only takes into account the expansion of water due to a temperature increase, and not other factors like polar ice caps melting, moving water into the ocean, which would logically only increase the rate at which the sea level rises, effectively decreasing the number of years required to rise the sea level by 3 metres.

*Mentor comment*

As an evaluation method the students looked up a prediction from a meteorologist (Kurt Wayne) to provide an external 'real world' estimate against which to test their outcome (they again omitted the web reference from their report). They identified an arithmetic error, caused by misreading expansion data, realised their predicted date was far too soon, and subsequently addressed the error. They revisited their original caveat concerning the possible impact of melting ice, coming up with a much longer timeframe – seemingly stretched by another arithmetic error. Calculations by the mentor based on data from the same website used by the team gave an answer of about 350 years from 1938 – interestingly within cooee of the Kurt Wayne prediction. Apart from the question of accuracy, the variations in the predicted timeframes served to illustrate why there is so much debate about the impact of climate change – itself a useful outcome. In various ways, the team can be seen to have considered all of the evaluation sub-criteria in their efforts to solve the problem chosen.

*Summary comment*

In considering the quality of the overall modelling report it is noted that the summary page is an IM²C requirement that did not apply in the present case. From the foregoing discussion and illustration, it is reasonable to say that all the characteristics listed above were displayed, and displayed consistently. Notwithstanding some procedural errors, and incomplete referencing, the reader has been given impressive evidence concerning modelling qualities of the team.

Finally, the purpose of this forensic examination is reiterated. The goal has not been to provide an assessment of a specific project as such, nor to display an 'ideal' solution, but to test the applicability and robustness of a set of criteria to a modelling project with no specific links to the IM²C. That is important for providing a basis for the evaluation of modelling projects, irrespective of their specific educational context.

In the following section, the criteria are applied with the specific intention of providing advice for attaining quality within the IM²C project.

# Relating modelling qualities to judging criteria

In this section, we draw on specific comments from IM²C judges and examples from various submitted IM²C solutions. These examples are taken from the teams that gained outstanding and meritorious awards in the International Mathematical Modeling Challenge. In general, we will not provide extracts here, but rely on the reader to find the IM²C entries via https://www.immchallenge.org.au/supporting-resources/previous-immc-problems

We discuss and illustrate qualities required in a modelling report to demonstrate how student reports can address the IM²C judging criteria. This will refer to the advice given to registered teams and their advisors as provided on the IM²C website as well as the guidance given to judges. The main aim of this advice is to help future Australian teams successfully tackle future IM²C problems.

Note that these cover all the points raised in this guidebook on the mathematical framework, the modelling process and writing a report; and that were used in the previous section on a modelling report illustrating use of criteria, to look at the structure of the team report on the 'Gold Coast inundated' problem.

The IM²C began in 2015, and Australian teams have participated in 2016 and 2017. To anchor the present discussion, we give:

- a reiteration of the Australian IM²C judging criteria
- the IM²C problem for 2015, along with some of the international judges' commentary
- the IM²C problem for 2016, along with some of the international and Australian judges' commentary, and further author commentary
- the IM²C problem for 2017, along with some of the international and Australian judges' commentary.

The Australian judging criteria is given on page 61.

# The International Mathematical Modeling Challenge (IM$^2$C)

The IM$^2$C is an international modelling competition involving teams of secondary students from a number of countries. The IM$^2$C poses a number of real-world mathematical scenarios, and each team works for several days using freely available material (from the web and other sources). At the end of this time, each team presents a report on their solution.

The challenge awards prizes to the top teams. The challenge has two levels: a national and an international level. On the basis of the results of the national competitions, the top two teams will be entered for judging at the international level.

The main aim of the IM$^2$C is to promote mathematical modelling, encouraging participants to explore the application of mathematics in real situations to solve problems of importance. Encouraging an extension of experience in mathematical modelling for students in secondary schools, the IM$^2$C seeks to develop and enhance students' ability to visualise, understand and apply mathematics in real-world contexts, providing a valuable opportunity for the practical demonstration of in-school learning and application of theory.

The point of involving teams of students is that difficult problems in society are almost always tackled by groups of people with different areas of expertise. The Challenge provides opportunities for students to get together to pool their knowledge and understanding for a common goal, to communicate, to reason and to solve problems. These are important factors of the Proficiency Strand of the Australian Curriculum and important skills in life.

Real-world problems require a mix of different kinds of mathematics for their analysis and solution, and take time and teamwork. The IM$^2$C provides students with a deeper experience both of how mathematics can explain our world and what working with mathematics looks like. By mobilising students in teams, the IM$^2$C replicates real-world conditions; requiring collaboration and contribution from different skill sets, perspectives and methodologies to achieve overall success.

Providing an opportunity for peer-based learning, this aspect of the IM$^2$C also helps to incorporate and reinforce the Australian Curriculum proficiency strands – understanding, fluency, problem-solving and reasoning – as students work together, communicate with one another and employ creativity, reason and logic to successfully solve the defined problem.

A fuller explanation of the instructions and rules for entry to the Australian challenge can be found on the website **www.immchallenge.org.au**

# 2015 IM²C: Movie scheduling


## Describe the real-world problem

A great deal of preparation must take place before a movie can be filmed. Important sets and scenes need to be identified, resource needs must be calculated, and schedules must be arranged. The issue of the schedule is the focus of the modeling activities. A large studio has contacted your firm, and they wish to have a model to allow for scheduling a movie. You are asked to answer the questions below. You should provide examples and test cases to convince the movie executives that your model is effective and robust.

### Question 1:

Develop a model that will produce a filming schedule given the following constraints:

- The availability dates of the stars of the film.
- The time required to film at a list of specific sites.
- The time required to construct and film on a list of sets.
- The availability dates for specific resources. For example a war movie might require helicopters which are available only at specific times.
- Some scenes cannot be shot until after certain computer generated content is defined and other physical items are constructed. Your schedule must include extra time to allow for redoing some shots if they turn out to be inadequate after editing and review.

### Question 2:

Develop a model that will take the information and schedule generated from the first question and can adjust them in the event that some delay in one aspect or the availability of some asset changes. For example, if one of the stars has an accident and cannot film for a certain period of time, you should be able to adjust the schedule.

### Question 3:

Use the model developed in the first question to develop a way to determine the most important constraints. That is, identify the constraints that will cause the longest delays if a problem occurs.

## 2015 international judges' commentary (edited extract)

What characteristics distinguished the better papers? The better teams developed and presented their models in a very logical manner. They moved from the very vague scenario with which they were provided to identifying a problem they could model mathematically. They explained their assumptions very clearly and discussed how well the assumptions were met by the situation they had identified. After analysing their model for solutions, they tested the model's conclusions against test cases they constructed or found in their research. They performed a sensitivity test to determine how the conclusions changed based upon changes in their data, thereby identifying the most important variables.

### Problem definition

Team 2015010 succeeded in profoundly developing the building of their model. Step-by-step, this team introduces the various elements of their model in a very clear manner that is understandable to a wide audience. Due to time restrictions, they did not succeed in extending this approach through the remainder of the paper, but clearly demonstrated the ability to present their ideas in a clear logical manner. For these reasons, paper 2015010 was awarded the meritorious award.

### Model formulation

The better papers typically developed an optimisation problem that generalised to larger data sets: more scenes, more filming sites, more actors with greater restrictions on their availability, and so forth. Typically, the algorithms for solving their optimisation models were computationally complex. The better papers used heuristics for finding good solutions in reasonable amounts of time, or adding constraints to their models to reduce the number of possible solutions. The team awarded the meritorious award (see solution 2015015) used both a heuristics-based algorithm and Kuhn's algorithm (also known as the Hungarian method) that guarantees a globally optimal solution and compared the results of the two methodologies. The better papers generalised to larger problems and tested their models using a case study or a practical example, and carefully presented their conclusions.

### Mathematical processing

A good example of sensitivity analysis can be found in solution 2015005, in which a great amount of random data was used to identify which variables were most sensitive. Another good example is in solution 2015004, in which the team provided a methodology for reacting to changes by choosing the objective of 'changing the current schedule as little as possible'. Both papers 2015004 and 2015005 stood out above the rest and were recognised as outstanding by the judges.

Use graphs, charts, networks and other appropriate visualisations where possible to aid understanding. A good example of using visualisations is provided by solution 2015017. The pictures of graphs included in their work clarify the complexity of the making of a film, thereby helping the reader understand the process that will be modelled in the following paragraphs.

### Model evaluation

A good example of testing a model can be found in solution 2015007. This team extracted data by hand from an actual movie, taking the data from the screen. This down-to-earth approach links the model with real-world data and gives a basis for testing and improving the modelling process. For these reasons, paper 2015007 was awarded the meritorious award.

### Report writing

The better papers excelled at scientific writing. The papers had a structure that was easy to follow. A very distinguishing discriminator among all papers was the quality of the summary. A good summary provides a very clear overview of what is accomplished in the paper.

An excellent example of a well-constructed summary can be found in solution 2015016. The summary gives an excellent overview of the work in a well-structured manner that invites the reader to read the complete work.

Our advice to future contest participants is to allow plenty of time to construct a report. Build a structure which allows you to present the development of your model in a logical fashion. Test your model in as realistic a fashion as possible, and clearly state your conclusions. Similarly, present the analysis and conclusions of your model in a manner that can be easily understood by a wide audience. Finally, write a summary that gives an overview of your work that excites the reader to study your paper.

Full judges' commentary and team solutions can be found on the website
**https://www.immchallenge.org.au/supporting-resources/previous-immc-problems**

# 2016 IM²C: Record insurance


Ethiopia's Haile Gebrselassie first won in 1994 and won for a third time in 2011. (Haile Gebrselassie at Vienna City Marathon 2011, Alexxx86/Wikimedia Commons)

## Describe the real-world problem

In athletics, one of the possible distances to run is 15 000 meters or 15k. For this type of run, 15k on a street track, there is a world record, as there are records for all other distances that are run in athletics (e.g., the marathon). In such a race, the organising committee will usually pay a significant amount of money as a bonus to the winner if he or she succeeds in setting a new world record. These amounts of money can get quite large in order to attract top runners: in the race shown in the picture there was a 25 000 euro bonus if the winner succeeded in improving the 15k world record – which, by the way, he (un)fortunately did not achieve. Had he done so, there would have been a major financial problem for the organising committee, since they had not purchased any insurance.

Usually, insurance will be purchased by the organising committee for such a running event, since the financial risks can be quite large. The fee they will have to pay for such insurance will be, of course, significantly lower than the bonus they would have to pay for a world record. Let's define the **average cost** of the bonus as the ratio of the amount of bonus divided by the expected number of times the event is replicated before the current record is broken. For example, if based on our analysis, we currently expect the record to be broken every 25 repetitions under conditions prevailing for a specified event, then the average cost of the bonus is 1000 euro per race. The first question is:

1. For the 15k run described above with a 25 000 euro bonus what is the average cost of the bonus?

The insurance company will add an amount to the computed average cost. The amount of the addition may be very reasonable or not. The insurance company expects to cover their costs and realise a profit over a long time period with multiple subscribers. The organising committee can decide to purchase the insurance or not (that is, 'self insure').

2. What criteria should the insurance company use in determining the amount to add to the average cost for the above race? Specifically, how do they weight each factor in determining their decision? For example, begin by considering the case where the insurer will add 20% to cover his operating costs, time value of money, and realise a profit over a period of time.
3. (a) What criteria should the organising committee use to determine whether or not they should purchase the insurance? Assume that they intend to sponsor this race many times in the near future. By self insuring, they expect to save the insurance company's added cost over a period of time.
   (b) But should they take the risk?

Now consider that you are a member of the organising committee of a major track meet with 20 men's and 20 women's athletic events, including field events (long jump, high jump, etc.).

**4.** Assume the organising committee can purchase the insurance or not for each of the 40 events. For example, they may choose to insure 10 of the 40 events. What factors should the organising committee consider in their decision to purchase insurance or not for each of the events at the meet? Specifically, how do they weight each factor in determining their decision?

**5.** Develop a general decision-scheme for the organising committees to determine for each event whether they should purchase insurance or self insure. This scheme should be written in a form easily understood and implemented by a typical organising committee.

Your submission should consist of a one-page Summary Sheet and your solution cannot exceed 20 pages for a maximum of 21 pages. (The appendices and references should appear at the end of the paper and do not count toward the 20 page limit.)

***Zevenheuvelenloop***

*Zevenheuvelenloop (Seven Hills Run in English) is an annual 15 kilometres road running race held in Nijmegen, Netherlands. It was first organised in 1984 and has grown to be one of the largest road races in the Netherlands; it attracted over 30,000 runners in 2008. The race has attained IAAF Label Road Race status. The inaugural edition of the race in 1984 featured only an 11.9 kilometre course as the Dutch athletics federation (Koninklijke Nederlandse Atletiek Unie) would not allow new races to be longer than 12 km. The current undulating, hilly course begins in Nijmegen, follows a path to Groesbeek and then loops back towards Nijmegen to the finish line. Zevenheuvelenloop lends itself to fast times: Felix Limo broke the men's world record in 2001 and, at the 2009 edition, Tirunesh Dibaba broke the women's world record over 15 km. In 2010 Leonard Komon improved Limo's still standing World Record.*

*A number of athletes have achieved victory at the Zevenheuvelenloop on multiple occasions; Tonnie Dirks, Tegla Loroupe, Mestawet Tufa, Sileshi Sihine and Haile Gebrselassie have each won the race three times. The 2002 winner, South African Irvette Van Blerk, won the race at the age of fifteen, having entered the race while holidaying in the Netherlands. The race was used as the test event for the development of the ChampionChip personal RFID timing system.*

*Source: Zevenheuvelenloop. (2017, September 19). Wikipedia, The Free Encyclopedia. https://en.wikipedia.org/wiki/Zevenheuvelenloop*

*Winners*

*Key: Course record*

| Edition | Year | Men's winner | Time (m:s) | Women's winner | Time (m:s) |
|---|---|---|---|---|---|
| 32nd | 2015 | Joshua Cheptegei (ETH) | 42:39 | Yenenesh Tilahun (ETH) | 50:05 |
| 31st | 2014 | Abera Kuma (ETH) | 42:18 | Priscah Jeptoo (KEN) | 46:56 |
| 30th | 2013 | Leonard Komon (KEN) | 42:15 | Tirunesh Dibaba (ETH) | 48:43 |
| 29th | 2012 | Nicholas Kipkemboi (KEN) | 42:01 | Tirunesh Dibaba (ETH) | 47:08 |
| 28th | 2011 | Haile Gebrselassie (ETH) | 42:44 | Waganesh Mekasha (ETH) | 48:33 |
| 27th | 2010 | Leonard Komon (KEN) | 41:13 WR | Genet Getaneh (ETH) | 47:53 |
| 26th | 2009 | Sileshi Sihine (ETH) | 42:14 | Tirunesh Dibaba (ETH) | 46:29 WR |
| 25th | 2008 | Ayele Abshero (ETH) | 42:17 | Mestawet Tufa (ETH) | 46:57 |
| 24th | 2007 | Sileshi Sihine (ETH) | 42:24 | Bezunesh Bekele (ETH) | 47:36 |
| 23rd | 2006 | Micah Kogo (KEN) | 42:42 | Mestawet Tufa (ETH) | 47:22 |
| 22nd | 2005 | Haile Gebrselassie (ETH) | 41:56 | Berhane Adere (ETH) | 47:46 |
| 21st | 2004 | Sileshi Sihine (ETH) | 41:38 | Lydia Cheromei (KEN) | 47:02 |
| 20th | 2003 | Richard Yatich (KEN) | 42:43 | Mestawet Tufa (ETH) | 49:06 |
| 19th | 2002 | Kamiel Maase (NED) | 43:41 | Irvette van Blerk (RSA) | 51:06 |
| 18th | 2001 | Felix Limo (KEN) | 41:29 WR | Rose Cheruiyot (KEN) | 48:40 |
| 17th | 2000 | Felix Limo (KEN) | 42:53 | Berhane Adere (ETH) | 48:06 |
| 16th | 1999 | Mohammed Mourhit (BEL) | 43:30 | Lyubov Morgunova (RUS) | 49:45 |
| 15th | 1998 | Worku Bikila (ETH) | 42:24 | Tegla Loroupe (KEN) | 50:06 |
| 14th | 1997 | Worku Bikila (ETH) | 42:20 | Catherina McKiernan (IRL) | 48:30 |
| 13th | 1996 | Josephat Machuka (KEN) | 43:06 | Marleen Renders (BEL) | 50:09 |
| 12th | 1995 | Josephat Machuka (KEN) | 42:23 | Hellen Kimaiyo (KEN) | 49:44 |
| 11th | 1994 | Haile Gebrselassie (ETH) | 43:00 | Liz McColgan (GBR) | 49:56 |
| 10th | 1993 | Khalid Skah (MAR) | 43:35 | Tegla Loroupe (KEN) | 50:06 |
| 9th | 1992 | Carl Thackery (GBR) | 43:54 | Tegla Loroupe (KEN) | 50:53 |
| 8th | 1991 | Tonnie Dirks (NED) | 44:09 | Ingrid Kristiansen (NOR) | 48:46 |
| 7th | 1990 | Tonnie Dirks (NED) | 44:53 | Carla Beurskens (NED) | 52:06 |
| 6th | 1989 | Tonnie Dirks (NED) | 43:31 | Carla Beurskens (NED) | 50:36 |
| 5th | 1988 | Robin Bergstrand (GBR) | 46:20 | Marianne van de Linde (NED) | 52:53 |
| 4th | 1987 | Marti ten Kate (NED) | 45:11 | Gerrie Timmermans (NED) | 57:16 |
| 3rd | 1986 | Sam Carey (GBR) | 46:2 | Denise Verhaert (BEL) | 53:33 |
| 2nd | 1985 | Klaas Lok (NED) | 45:28 | Joke Menkveld (NED) | 57:28 |
| 1st | 1984 | Leon Wijers (NED) | 36:55 | Anne Rindt (NED) | 45:48 |

*Winners by country*

| Country | Men's race | Women's race | Total |
|---|---|---|---|
| Ethiopia | 11 | 12 | 23 |
| Netherlands | 7 | 6 | 13 |
| Kenya | 9 | 7 | 16 |
| United Kingdom | 3 | 1 | 4 |
| Belgium | 1 | 2 | 3 |
| Ireland | 0 | 1 | 1 |
| Morocco | 1 | 0 | 1 |
| Norway | 0 | 1 | 1 |
| Russia | 0 | 1 | 1 |
| South Africa | 0 | 1 | 1 |

## 2016 international judges' commentary (edited extract)

### Problem definition

The better papers resulted as teams developed and presented their models in a very logical manner. We strongly suggest that teams clearly answer each specific question in the problem. The use of headers helps the judges understand which aspect of the given problem the team is currently addressing.

Meritorious paper 2016036 is an example of presenting a model in a very logical and reasoned manner. Each question is addressed by reference to the model they developed from their assumptions.

### Mathematical processing

Meritorious paper 2016004 is an example of excellent mathematics but needed greater attention by the authors to their explanation of why the mathematics is appropriate to a model developed from their assumptions, even though their examples are quite good.

### Model formation

There was great variety in modelling the probability of setting a world record at a particular event in a given track meet. Some papers used a simple approach estimating the trend in time intervals between world records. Others considered the historical records of specific athletes and determined the probability of a specific athlete breaking the current world record. Those athletes who had a reasonable chance of breaking a record were then placed in a group of 'top athletes'. They then considered the number of top athletes attending an event and the probability that none of these athletes would break the record. Some weighted recent data more heavily and examined the recorded times of the athlete versus their age and other variables.

Paper 2016021, which was awarded the designation of outstanding, uses a simple approach to the probability by estimating the number of races before a record is broken. Paper 2016026, which was also awarded the designation of outstanding, examines particular athletes and computes the probability of that athlete breaking the record based upon his recent performances in the event and the current world record. They then compute the probability that no top performer attending the race will break the record. Paper 2016033, which was awarded the designation of outstanding, tested two approaches. The first is based upon assuming that participants have equal probability to break the record each year. The second is based upon assuming the world record will be broken at an interval within a range. After testing, they chose the second approach.

A wide variety of approaches were used to model the decisions of the insurance company: how would they price the insurance premium to cover their operating expenses, make a profit exceeding alternative investments, yet at a price attractive to organising committees?

Meritorious paper 2016015 presents a unique method that is reasonable to employ. This team also excels at communicating ideas and justifying assumptions.

We saw many approaches to the decision of the organising committees. What were the short- term and long-term goals of the committee? What long-term and short-term risks were involved in accomplishing their goals if they did not buy the insurance? What does 'bankruptcy' mean for a particular organising committee?

Papers 2016021, 2016026, and 2016033 were all awarded the designation of outstanding and employ distinct objectives for the weighting of short-term and long-term risks and the weight placed on avoiding bankruptcy. They then develop decision models to achieve their stated objectives.

### Model evaluation

It is important to address the strengths and weaknesses of the model you construct. All models have limitations and it is important that you recognise and state those limitations.

We strongly recommend that you read paper 2016026, which was awarded the designation of outstanding, and demonstrates the use of mathematics appropriate to the team's assumptions, combined with some very good writing skills.

### Report quality

Our advice to future contest participants is to allow plenty of time to construct a report. In fact, consider working on the report as soon as you begin work on the problem, as communicating your ideas and approach is critical in this challenge.

The better papers demonstrated excellent writing, particularly in the quality of the summary. An example of an excellent summary can found in the meritorious paper 2016025. The summary gives an excellent overview of the work in a well-structured manner that invites the reader to read the details of the complete work.

The best papers presented the analysis and conclusions of the model in a manner that could be easily understood by a wide audience. Consider who will use the model you have built and explain your model to that audience as well as to the judges. Use graphs, charts, networks and other appropriate visualisations where possible to aid understanding. An excellent example of incorporating graphs intelligently and conveniently for the reader is contained in meritorious paper 2016025.

Consider who will use the model you have built and explain your model to that audience as well as to the judges.

Remember that the judges are all from different countries. Explain your work using universally understood language. Also, the judges are not familiar with the curricula of each school district. Thus, you should build a structure which allows you to present the development of your model in a logical and easily understood fashion.

The judges are not looking for the papers using the most sophisticated mathematics – this is a mathematical modelling competition. Rather, they expect you to use the mathematics you already know in a logical manner to reach conclusions from your assumptions. Typically, simpler is better.

Full judges' commentary and team solutions can be found on the website
**https://www.immchallenge.org.au/supporting-resources/previous-immc-problems**

## 2016 international judges' commentary (edited extract)

### Problem definition

An essential early step in working on a modelling problem is to decide what the problem is really about. Sometimes this is not obvious.

For the 2016 problem, scaffolding was provided in the form of specific questions – these were presented to guide teams through the stimulus material and to help them make worthwhile progress towards the desired end result – as well as a dataset. However, this scaffolding may have made some teams think that the main problem was to answer these questions. In fact, a better approach would have been to see those as stepping stones towards the problem as articulated in the final question, which was about formulating well-argued and well-supported advice to the race organising committee about how it should handle the risks associated with offering incentive payments to competing athletes across a number of events.

### Model formulation

The best models were clearly identified, and the variables used in the models were fully explained. The assumptions underlying the models used were also identified, explained and justified. In solving real-world modelling problems, it is usually necessary to impose assumptions in order to simplify the situation sufficiently to make progress towards a solution.

In addition, the choice of data values to test, demonstrate or to justify the models used is a significant part of the modelling process, and therefore also warrants solid argument and justification as to the choices made.

Testing the sensitivity of a model is also an important step in developing a model that will function as desired in the situation for which it is built. Using a range of parameter values to evaluate this sensitivity is often a very useful activity.

### Mathematical processing

The particular mathematical tools that could be applied to any modelling task can vary enormously. Sometimes a simple piece of mathematics can be all that is needed to analyse some aspect of the problem situation. The level of complexity of the mathematics used is not necessarily a critical factor, rather it is the relevance, appropriateness and usefulness of the mathematics that is most important, and its interpretation in relation to the situation under study. The key factor is the fitness for purpose of the mathematics used.

For example, a graph of data that shows a relationship that is not linear can provide a clue that a non-linear model for the relationship should be explored. It would not matter how well a linear regression model fits if it is not at all appropriate for this particular problem.

The nature and level of mathematical content of good modelling reports can vary, and the relationship between the level and complexity of mathematics used to the overall quality of the report is also not direct. Use of simple mathematics presented as part of a well-argued and clearly described solution can be far more effective than attempts to use more advanced techniques that are poorly explained or justified.

Whatever the level of mathematics used, though, it should be used correctly, and this feature of a mathematical modelling report will stand out very clearly to reviewers, particularly because they will have a mathematical background themselves. For example, if a report refers to 'equations' that are missing one or more of a left-hand-side, an equals sign, or a right-hand-side (and so are expressions rather than equations), this will not be viewed positively. Similarly, if a formula from a book or article is used, it must be transcribed and applied correctly.

The use of mathematical tools, especially technological tools such as programmable calculators, or the use of statistical software, can also be a help as well as a hindrance. Wise decisions about the use of technology will stand out in a well-considered piece of modelling, as will poor decisions. Graphing tools can be very powerful aids to investigating relationships among variables, and statistical techniques can be very helpful in establishing best fit relationship models. But including many colourful graphs that serve no real purpose will generally not contribute positively to a modelling report. Wherever possible, predictions made from the models applied should be checked against available (or constructed) data; and all mathematical work undertaken should be interpreted in relation to the problem under study.

It should be obvious to a reader why a particular mathematical tool or technique was chosen as the most useful in the circumstances. This will usually warrant explanation and justification, and results from using that tool should be interpreted with clear reference to its relevance to the purpose of the work.

### Model evaluation

An essential element of a good modelling effort involves thorough evaluation of the model. Does it give information needed to make decisions about the questions at issue? To what extent would changes in the assumptions made, and changes to the settings assumed in the way the problem is specified, affect the usefulness of the model? Sensitivity analysis and other forms of model evaluation can be very challenging, but they must be pursued.

Model evaluation can be undertaken by varying the numeric values that could apply in the situation under study, including by finding suitable data from relevant sources. In some circumstances, this can be done systematically through simulating data, for example where historical data cannot easily be obtained.

Full judges' commentary and team solutions can be found on the website
**https://www.immchallenge.org.au/supporting-resources/previous-immc-problems**

### Report quality

A good quality report of the outcomes of a modelling task can take a variety of forms, and the most important feature of a high quality report is that it meets the need at hand. In the case of the 2016 IM²C problem, for example, the context of the problem statement provided the perfect opportunity to imagine the organising committee as the key audience for a piece of written advice as to what they should do regarding managing their event, attracting high-level performers, and managing the associated risks through their insurance decisions. Such a report may have contained some mathematics, but only to the extent that it would help the organising committee understand the advice and reasons for it.

A good modelling report does not look like a mathematics assignment. It is not a set of results as if it were a response to a test or a structured assignment. It's a piece of analysis and discussion that meets the needs of the particular situation under study. Detailed calculations and other mathematical content would often best be presented as an attachment or appendix to the main report.

In writing a summary, the goal should be to engage the reader and to draw them in to the analysis provided. In the 2016 case, an 'executive summary' could have been a useful model for the report summary.

### Further commentary on 2016 solutions

In this table, we look at examples to illustrate the points that the IM²C Australian judges discussed. These examples are taken from the teams that gained outstanding and meritorious awards in the 2016 IM²C.

*Australian judges' notes on approaches to the 2017 IM²C problem, by judging criteria*

| **Criterion 1: Problem definition** |
|---|
| **Specification of precise mathematical questions from the general problem statement** |
| See 2016026 as well as 2016021 in, for example, the start of part 2. |
| **Criterion 2: Model formulation** |
| **Identification of assumptions with justification** |
| Teams 2016021, 2016025 and 2016026 provide good descriptions and justifications of their assumptions. The first two teams do this in prose, while the third team uses a table. |
| **Choice of variables** |
| See 2016036 for a good example.<br>Team 2016021 lists the main variables used at the very start of the report. The reasons for these choices appear as the report unfolds. |
| **Identification and gathering of relevant (needed) data** |
| Team 2016025, in Graph 2, shows technology output. It shows additional data represented as a graph showing as points the marathon records since 1908. (Note that the am times for the vertical axis should be hour times.) This graph appears to have been produced by the use of a commonly available piece of software.<br>Team 2016026 collects data on past world records (see the solution's section 1.7).<br>Team 2016036 produces a frequency distribution table from accessing the Zevenheuvelenloop data from the 26th to the 32nd race data.<br>Team 2016044 uses the 100 metres, 10 000 metres and high jump world records. This was so that it could predict various world records and then apply their model to the problem in hand. This enabled the team to minimise the effect of things such as 'technological advancement, development in training practices and anti-doping laws'. |
| **Choice and justification of parameter values** |
| See the section on 'the amount of the insurance premium' in solution 2016025. |
| **Development of mathematical representations** |
| See, for example, the section on 'the amount of the insurance premium' in 2016025, and solution 2016021, part 2. |

## Criterion 3: Mathematical processing

### Application of relevant mathematics

A variety of mathematical techniques and ideas are used by different teams in the attempts to find solutions to the questions. These include:

- graphs (2016025)
- probability (2016026)
- regression (2016026)
- curves of best fit (2016025)
- benchmark percentage (2016026)
- product and summation notation (2016026)
- calculus (2016026, 2016036 - integration)
- labelled graphs or networks (2016026)
- expected values (2016026)
- matrices (2016026 – eigenvalues)
- statistics (2016004, 2016021, 2016036 – Pareto distribution, extreme value theory, binomial distribution, confidence intervals)
- Excel spreadsheets (2016004).

### Invocation and use of appropriate technology

This is done by 2016021 by using Monte Carlo Simulation in part 3 of the report. Given the importance of technology in modelling, it would be good to have a few more varied examples of technology use.

### Checking of mathematical outcomes for procedural accuracy

This was consistently done for each of the models produced by team 2016036 (see sections 2.4, 3.4 and 4.4).

See also the 'example case' in parts 2 and 3 of 2016021.

### Interpretation of outcomes in terms of the problem situation

Page 4 of team 2016044's work uses consecutive years that were and were not followed by a world record in the 10 000 metres race. On the basis of this, the team was able to set limits for the range of probabilities that a world record would or would not succeed a given period of years. This was then fed into their decision process for the organising committee.

## Criterion 4: Model evaluation

### Adequacy and relevance of findings in relation to problem situation

See team 2016021 in 'strength and weaknesses' in part 5 of their work; team 2016015 in 'strengths and weaknesses' on page 5; and team 2016036, in, for example, section 4.5.

### Further elaboration or refinement of problem

Team 2016021 approaches the first question in two ways: first through direct computation and then using confidence intervals.

Team 2016033 uses two approaches to Question 3.

### Relevance of revised solution(s) following revisiting and further work within earlier criteria

2016026 follows two models through together for the first section. These were produced by different approaches and compared at the end of the first section.

### Quality of answers to specific questions posed in problem statement

2016044 discusses the advantages and disadvantages of the team's world-record determining model at the end of stage 4.

2016021 notes on page 5 of 20 that the error increases as the number of estimations increases.

## Criterion 5: Report quality

### Summary page quality: succinctness, power to attract reader

We strongly recommend reading the summary of team 2016025 in full because the international judges' make specific reference to it as an excellent example. This report is clearly well written and provides material of interest that is easy to read and able to be understood by the recipients, the organising committee of the 25km race.

### Overall organisation of the report

Solution 0216036 was praised by the judges for it being 'an example of presenting a model in a very logical and reasoned manner. Each question is addressed by reference to the model they developed from their assumptions.'

### Logical presentation including:

- description of the real-world problem being addressed

Team 2016026 provides a 'problem restatement' almost immediately after the summary. This appears as a list.

Team 2016021 includes an introduction that reorganises and expands the ideas inherent in the various questions.

- specification of the mathematical questions

Team 2016021 specifies mathematical questions throughout its work.

- listing of all assumptions wherever they are made

2016026, for example, introduces new assumptions as they are needed at the start of answering each question. The team though does provide 'overall assumptions and justifications' immediately after the 'problem restatement'. In all cases the assumptions and justifications are presented in table form.

- indication of how numerical parameter values used in calculations were decided on

This can be found in all reports.

- setting out of all mathematical working: graphs, tables; technology output and so on

Graph 2 from team 2016025 is technology output. The curves are curves of best fit, one (blue) modelling the marathon record times up to the late 1960s and the other (red) for the last 40 years. Before this graph there is a clear explanation of the graph and its potential use in the Project.

Team 2016033 produces specifically designed programs to facilitate calculations (see, for example, Approaches 2.1 and 3.2.2). Team 2016021 also does this in their appendix on MATLAB programs.

Excel spreadsheets were used by 2016044 to fit data distributions and make calculations easy.

- interpreting the meaning of mathematical results in terms of the real world problem

This is done well by all teams.

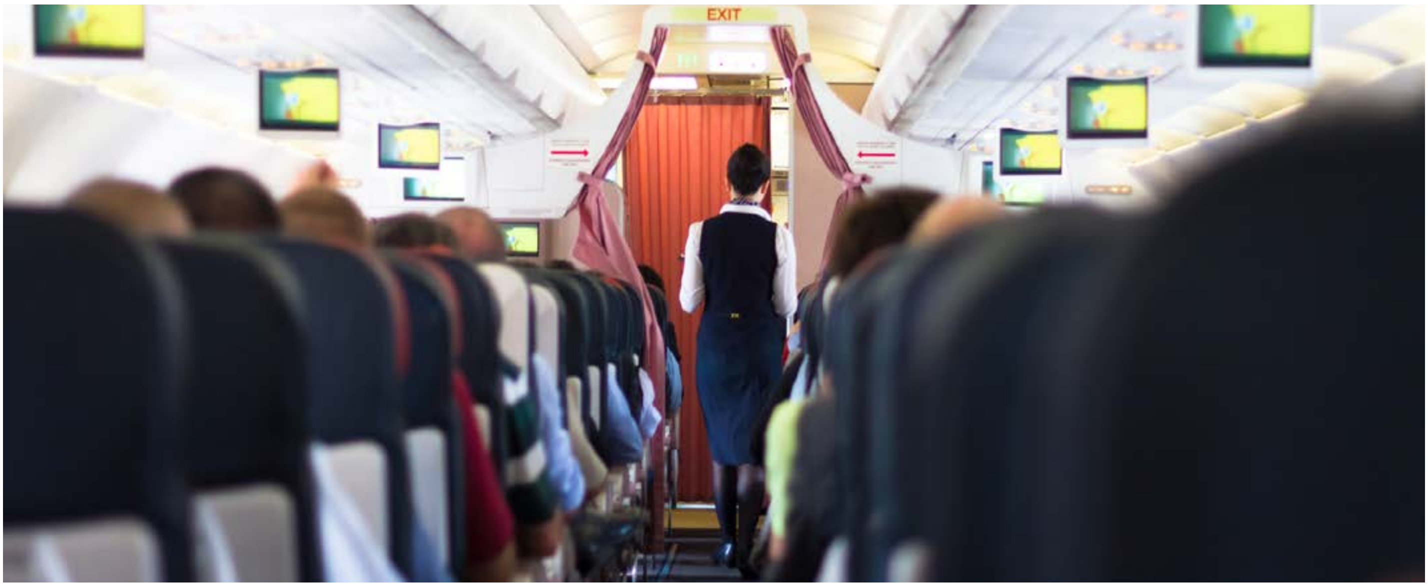
- evaluating the solution(s) in terms of the problem requirements

2016026 discusses the limitations of its model, including the large number of variables that the team has needed to use. The team is also concerned about the usability of its model without a computer.

2016044 uses spreadsheets do calculations and so makes the work within the capabilities of the organising committee.

It is worth noting that, of the seven team's work considered here, only one provided simple instructions for the organising committee in their conclusion section. The instructions were clear to the teams but generally required some quite sophisticated mathematical knowledge to be able to calculate the unknowns and make decisions as to whether to insure or not. Team 2016015 produces a 'decision-making scheme', but it requires the organising committee to be able to find the probability of a world record, as well as a geometric mean. The closest any team appears to have got was 2016021 who provide computation device they call a 'risk calculator' for Question 3.

# 2017 IM²C: Jet lag




Organising international meetings is not easy in many ways, including the problem that some of the participants may experience the effects of jet lag after recent travel from their home country to the meeting location which may be in a different time zone, or in a different climate and time of year, and so on. All these things may dramatically affect the productivity of the meeting.

The International Meeting Management Corporation (IMMC) has asked your expert group (your team) to help solve the problem by creating an algorithm that suggests the best place(s) to hold a meeting given the number of participants, their home cities, approximate dates of the meeting and other information that the meeting management company may request from its clients.

The participants are usually from all corners of the Earth, and the business or scientific meeting implies doing hard intellectual team work for three intensive days, with the participants contributing approximately equally to the end result. Assume that there are no visa problems or political limitations, and so any country or city can be a potential meeting location.

The output of the algorithm should be a list of recommended places (regions, zones, or specific cities) that maximise the overall productivity of the meeting. The questions of costs are not of primary importance, but the IMMC, just as any other company, has a limited budget. So the costs may be considered as a secondary criterion. And the IMMC definitely cannot afford bringing the participants in a week before the meeting to acclimatise or give them the time to rest after a long exhausting journey.

Test your algorithm at least on the two following datasets.

### Scenario 1: Small meeting

Time: Mid-June

Participants: 6 individuals from:

- Monterey CA, USA
- Zutphen, Netherlands
- Melbourne, Australia
- Shanghai, China
- Hong Kong (SAR), China
- Moscow, Russia

### Scenario 2: Big meeting

Time: January

Participants: 11 individuals from:

- Boston MA, USA (2 people)
- Singapore
- Beijing, China
- Hong Kong (SAR), China (2 people)
- Moscow, Russia
- Utrecht, Netherlands
- Warsaw, Poland
- Copenhagen, Denmark
- Melbourne, Australia

Your submission should consist of a one-page summary sheet. The solution cannot exceed 20 pages for a maximum of 21 pages. (The appendices and references should appear at the end of the paper and are not included in the 20-page limit.)

## 2017 international judges' commentary (edited extract)

### Problem definition

The better papers moved from the rather vague scenario that required maximising productivity to identifying a problem they could model mathematically. Paper 2017045 shows a very consistent, clear and pleasantly readable approach of the problem, even though it may to be a rather simple one. This paper shows that even without using complicated formulae it is possible to describe a good model. Furthermore, in both paper 2017019 and paper 2017026, a nice use has been made of various types of pictures to explain their method and thinking, thus avoiding heaps of text and formulae, thereby making the reading much more easy and pleasant.

### Model formulation

Teams developed a wide range of definitions of productivity. First, they identified variables, such as 'jet lag' caused by traveling in an east-west direction (typically approximated by the number of times zones crossed), the distance travelled by each attendee, the total travel time (including layovers) of each member, the ability of the destination city to host the meeting, changes in sunset time for each attendee, climate conditions especially temperature and altitude, and other factors. Next, there was great variation on forming an objective function.

Some teams picked a single factor such as jet lag to optimise. Some teams picked several factors and weighted the relative importance of each factor to form the productivity function. Several teams picked the variables they wished to consider and addressed them sequentially. For example, Team 2017045 outlined a process of first determining a time zone in which to meet; then, within that time zone, the general area that minimised the average travel distance and travel time; then, within that area, an acceptable climate zone; then, within that zone, the cities appropriate to host the meeting. By considering the most important variables first and allowing an acceptable range of satisfaction, they developed a process for converging to a solution they liked.

Another distinct approach was to research a list of ideal cities to host meetings. Students found different organisations which ranked the desirability of the best host cities based upon different criteria. They then began eliminating cities based upon one or more of the variables listed, such as jet lag, distance travelled, time of travel, and so on.

After defining productivity using a principal variable or a subset of variables, another important decision needed to be addressed. For example, suppose jet lag were the only variable considered.

Should we minimise the average amount of jet lag? For example, if Shanghai were chosen, the average time zones crossed would be minimised. But one of the six members (Monterey, USA) would have to cross nine time zones. And the member from Melbourne, Australia would have a great distance to travel and a long travel time. What is the effect on the six-person group productivity of one or more members with low productivity as defined by your team? Would a better solution be to minimise the maximum number of time zones crossed by a member of the group, or minimise the maximum distance travelled, or minimise the maximum time of travel of any member of the group?

### Mathematical processing

This is a mathematical modelling challenge, so solutions should not force the mathematics upon the given scenario. Rather, a team should begin with the simplest mathematics that solves the problem the team has identified. Later, if you feel it is appropriate, you can refine your model to increase the accuracy, or change your assumptions to find a more appropriate solution.

## Model evaluation

There were some common shortcomings among solutions that could have been addressed through more rigorous model evaluation.

Some teams did not consider travel time in realistic settings. Many teams used distance between cities as an approximation, which is not a good assumption. As a result, those teams computed absurd answers such as the middle of Siberia and North Korea. In reality, considering actual travel time makes the problem much easier. The meeting will preferably be held in cities with international connections or towns that are easily accessible from such cities.

Many students used temperature difference as a penalty. But in reality, even people from very cold areas in winter would not mind being in a place like San Diego in winter, even though the temperature difference is huge.

Some teams over-emphasised the difference between east-west vs west-east travel. Actually, east-west long distance travel is typically faster because of the jet stream. Hong Kong to Seattle is 11 hours while Seattle to Hong Kong is 13.

### Report writing

The summaries of the better papers were excellent. Examples of good summaries can be found in papers 2017007 (a clear explanation of the process followed by the team), 2017020 (with a nice description why some aspects were not taken into account) and 2017054 (very 'to the point').

Outstanding paper 2017057 presents a comprehensive and appropriate mathematical model, clearly communicates the team's modelling process, and articulately explains their results, as well their model's strengths and weaknesses. These characteristics, in general, distinguish better papers.

Pictures, especially graphs, tables and schedules, can explain quite efficiently your ideas and by using them you can sometimes avoid a lot of text. Also the use of relevant pictures and graphs makes a report clearer and more pleasant to read. But please realise sometimes tables might better be in an appendix. The use of formulae is of course quite essential in a mathematical modelling assignment. But the use of unexplained formulae will not make the report more convincing. So always make sure the reader believes the formulae used are at least understood by the writers themselves. Also realise the readers of your report are, though experienced mathematicians, not experts in all parts of the great world of math!

Appendices, especially references, are very useful. But do not expect the judges to read your appendices. We may refer to an appendix to check a reference or a step in your computer code, but otherwise may not examine or read an appendix. So, do not place anything important to the development of your model in an appendix.

## 2017 international judges' commentary (edited extract)

### Problem definition

An essential starting point is to clarify exactly what the IM2C problem requires. The problem statement asks teams to develop an algorithm, and to test that algorithm on at least two given scenarios. The question does not ask where those two meetings should be held, but asks for a systematic process (an algorithm) that could be followed to determine a location for a meeting of participants from different home locations that maximises productivity of the participants, especially in relation to the effects of jet lag.

Another essential step in an effective modelling activity is to transform the statement of what is required into a mathematical objective. As well as developing a clear understanding of what would constitute an answer to the question in its context, the goals of the exercise must also be expressed in clear mathematical terms. For example, the mathematical objective could be to minimise total distance travelled by the meeting participants, or time spent travelling, and so on.

### Model formulation and mathematical processing

In between the beginning process of defining the goals of the task and defining that in mathematical terms, and the end process of writing a report, the processes of model formulation, mathematical processing, and model evaluation take place. Those processes would often occur multiple times, since the first attempt to solve the problem might expose other issues to be taken more into account to provide the best possible solution. A better mathematical formulation might be needed, which might require an adjusted model that a different kind of mathematical processing could assist, and an updated interpretation of the results and evaluation of the outcomes would then be needed.

An example taken from the 2017 IM2C reports was from a team that did an excellent job of defining the cumulative effects of jet lag across the meeting participants as the total number of time zones needed to be crossed to get to the meeting's location. After this team established that mathematical objective, team members then systematically tested each potential location time zone to determine the total number of time zone changes that would occur. This enabled them to narrow their solution to the time zone that created the fewest time zone changes for all participants, which is where their investigation stopped. Another iteration of the modelling process for that team might have been to then apply other factors, such as existing flight routes, or climate factors, to further narrow their recommendation.

### Model evaluation

No modelling process is complete without an evaluation of the solution proposed. Does it answer the question? How would a change in the assumptions or starting conditions affect it? What additional factors could be taken into account to make the solution work in a wider variety of circumstances? Very few of the IM2C 2017 reports considered the extent to which their solution was 'best' or what other possible solutions might have been equally or almost as good. Very few teams showed that their solution would apply to completely different scenarios from the two cases given as part of the problem statement.

### Report writing

The end-point of the modelling process is to communicate the results in a form that can be understood and used by its audience. The report required for the IM2C comprises three parts: a one-page summary sheet, a report of the solution, and appendices, which includes references. However, the exact way a report is constructed should be determined in light of its purpose and audience. A modelling report is not the same as a school mathematics assignment. It might take the form of a recommendation or set of recommendations to a committee, together with an explanation and justification of the recommendations.

| Better approaches | Problematic treatments |
| --- | --- |
| **Model formulation** | |
| *Identifying relevant variables* | |
| ■ Recognise that meetings are usually held in climate-controlled buildings.<br>■ Seek to incorporate climate in recognition of out-of-meeting activities.<br>■ Use scientific data to optimise working conditions (choose 'best' latitude with defensible definition of 'best'). | ■ Look for an 'average' climate – ignores that people react differently to their usual climate, to variations in climate, and to shorter-term weather changes.<br>■ Fail to distinguish between weather and climate. |
| *Identifying assumptions and other factors* | |
| ■ State assumptions clearly and explain why they are made (e.g. to simplify the problem).<br>■ Show how the assumptions contribute to the solution path followed.<br>■ Consider the possible impact of changing assumptions.<br>■ Factors used are justified (e.g. evidence cited) and linked to solution.<br>■ Impact on cost of using home location of a participant. | ■ Making pointless, unrealistic, or unfair assumptions (e.g. no flight delays, no crying babies will be on the flight, all meeting participants are in good health to minimise health-related exacerbation of jet lag).<br>■ Factors simply stated with no justification or evidence, and no link to solution.<br>■ Extensive exploration of flight costs, hotel costs, meeting room hire costs etc. |
| **Mathematical processing** | |
| *How 'distance travelled' was treated* | |
| ■ Consider actual flight arrangements (such as proximity to airport, existence of direct flights, total travel time) | ■ Treat all distances 'as crow flies' rather than actual journeys required, including great circle calculations that don't take actual flight routes into account, or using three-dimensional coordinates only.<br>■ Ignore multiple participants from particular origins |
| *How time zones were treated* | |
| ■ Seek to minimise the total number of time zone changes for participants.<br>■ Note that the average UTC offset (or equivalent) does not necessarily minimise total time zone changes.<br>■ Take account of changes in time zone differences at different times of the year.<br>■ Quantify the daily period of alertness (and its overlap) for participants (and therefore spell out the impact of jet lag).<br>■ Recognise jet lag diminishes each day.<br>■ One report defined a productivity function, applied it to participants according to their 'normal alert hours', and integrated it to find the total work achieved. | ■ Only time zone considered without taking account of actual journeys.<br>■ Time zone calculations performed without checking the total time zone changes that result (e.g. note that the 'average time zone' calculation does not necessarily yield the location with the least number of time zone changes)<br>■ Ignore multiple participants from particular origins.<br>■ Ignore the effect of daylight saving on time zone data.<br>■ Assume the days required for jet lag recovery can be added to the pre-meeting time (contradicts problem statement). |
| **Model evaluation** | |
| ■ Consider the possible existence of multiple ideal locations.<br>■ Consider the range of different locations that could provide more or less equivalent solutions.<br>■ Consider the applicability of the algorithm for other scenarios (eg, different kinds of configurations of origin locations – such as several participants coming from particular region, with only one or two coming from different region) | ■ Find just one proposed city for each scenario.<br>■ Propose a solution that does not pass the laugh test (eg, clearly looks wrong from inspection of maps provided; is in the middle of no-where) |

# Additional example problems from the IM$^2$C

The International Mathematics Modeling Challenge has been run annually since 2015. A number of other contests have led up to the establishment of IM²C. The Consortium for Mathematics and its Applications has long run the High School Mathematical Contest in Modeling, in the United States of America. Several sample prompts from this contest are outlined below.

## Bank service

The bank manager is trying to improve customer satisfaction by offering better service. Management wants the average customer to wait less than two minutes for service and the average length of the queue (length of the waiting line) to be two persons or fewer. The bank estimates it serves about 150 customers per day. The existing arrival and service times are provided. Determine if the current customer service is satisfactory according to the manager's guidelines. If not, determine, through modelling, the minimal changes for servers required to accomplish the manager's goal.

## Emergency medical response

The Emergency Service Coordinator for an area is interested in stationing the three available ambulances to provide the best likely medical response. Area maps, population data and average travel times are provided. Determine the best locations for the three ambulances to maximise the number of people who can be reached within 8 minutes of an emergency call.


## American elk

Prior to the arrival of European colonisation on the North American continent, the ecological bio-diversity was much richer than we currently know in the 21st century. The elk native to the eastern areas of the United States was hunted to extinction sometime in the early 1800s. A similar species, the Manitoba Elk, adapted to living in the western US and Canadian prairie, developing tolerances to certain diseases, foods, and environmental differences. How would these adaptations affect an introduction of Manitoba Elk into the eastern states? Population data from a reintroduction trial are provided. Build a mathematical model to determine whether the elk survive or die out. Come up with a plan to improve the growth of the elk population over time.

## Fuel prices

Fuel prices fluctuate significantly from week to week. Consumers would like to know whether to fill up the tank (fuel price is likely to go up in the coming week) or buy a half tank (fuel price is likely to go down in the coming week). Develop a model that a consumer could use each week to determine how much fuel – full tank or half tank – to purchase.

## Space shuttles

On July 21, 2011, the 135th and final United States Space Shuttle landed in Florida after its 13-day mission into orbit, complete with a docking at the International Space Station. The National Aeronautics and Space Administration will now have to rely on other nations or commercial endeavours to travel into space until a replacement vehicle is developed and constructed. Develop a comprehensive 10-year plan complete with costs, payloads, and flight schedules to maintain the International Space Station.


## Search and find

Finding lost objects is not always an easy task, even when you have knowledge of a general location. Consider the following scenario: you have lost a small object, such as a class ring, in a small park. A map of the park is to be provided. It is getting dark and you have your pen light flashlight available. If your light shines on the ring, you assume that you see it. You cannot possibly search 100 per cent of the region. Determine how you will search the park in minimum time. An average person walks approximately 4 miles per hour. You have about two hours to search. Determine the chance you will find the lost object.

## Bicycle club

Several cities are starting bike share programs. Riders can pick up and drop off a bicycle at any rental station. These bicycles are typically used for short trips within the city centre, either one-way or roundtrip. The idea is to help people get around town on a bike instead of a car. Those making longer trips (such as commuting to work) are likely to use their own bikes. Some of the challenges are how to determine where to locate the rental stations, how many bikes to have at each station, how/where to add new locations as the program grows, how many bikes to move to another location and when (time of day, day of week). The downtown city maps, the bike rental locations and the number of bikes at each location for several are available online. Develop an efficient bike rental program for these cities.

## Curbing city violence

A regional city has had lots of problems with gangs and violence over the years. The mayor, chief of police, and city council need your help. Data are available for the following: incidents of violence, homicides, assaults, regional population (census data), unemployment, high school enrolment, high school attrition, graduation rate, prison population (adult and juvenile), released on parole, parole violations. Analyse and model these data to give the city a plan to reduce violence. Prepare a report for the mayor outlining your proposals.

Full problems with example data and possible solutions can be found on the website **https://www.immchallenge.org.au/additional-example-problems**