THREE TYPES OF TASK
- EXTENDED INVESTIGATIONS
- IN-CLASS INVESTIGATIONS
- INVESTIGATIVE QUESTIONS

18 ASSESSMENT TASKS

INCLUDES SOLUTIONS WITH MARKING KEYS SHOWING MATHEMATICAL BEHAVIOURS

WRITTEN BY TEACHERS FOR TEACHERS

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AC Australian CURRICULUM

The Mathematical Association MAΣWA

Free Sample set of investigative questions

INVESTIGATION ASSESSMENT TASKS YEAR 11 SPECIALIST MATHEMATICS

Contents

SPECIALIST MATHEMATICS: Year 11

TASK 9

Investigative questions

Unit 1

Topic 1.3: Geometry

Course-related information

The concepts and skills included in this investigation relate to the following content descriptions within the Australian Curriculum Specialist Mathematics syllabus:

- use proof by contradiction (ACMSM025)
- The angle at the centre subtended by an arc of a circle is twice the angle at the circumference subtended by the same arc (ACMSM030)
- The alternate segment theorem (ACMSM034)

The ability to choose and use appropriate technology to enhance and extend concept development is also required for some of the items.

Background information

The ability to prove two triangles are congruent is assumed together with the knowledge of the exterior angle of a triangle theorem, the central angle theorem and the angle in the alternate segment theorem. It is also assumed that students are familiar with the method of proof by contradiction (Specialist Mathematics Unit 1 Topic 3: Geometry). Each of the questions provided is stand-alone and may be incorporated into different tasks: it is not intended that they form part of the same assessment task.

Task conditions

These questions are independent of each other and are written so that they may be incorporated into a test or examination. Student access to a graphical/CAS calculator is assumed. The time required to complete each question is left to the discretion of the teacher but the intention is that each question may be completed within 15 minutes. In question 3, students could be asked to express θ in terms of α and/or β for any of the diagrams, three of which have been included here.

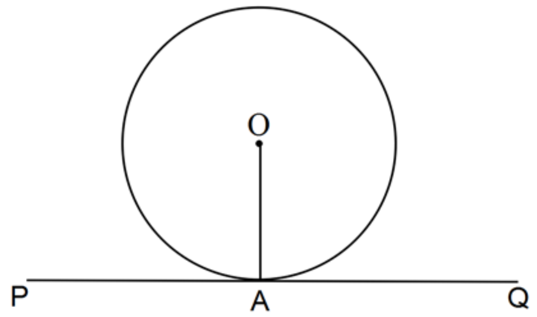
Acknowledgement

Question 3 was based on the article Circles and the Lines That Intersect Them, Ellen L. Clay and Katherine L. Rhee, appearing in the NCTM journal Mathematics Teacher, December 2014/January 2015.

Investigative questions for Topic 1.3

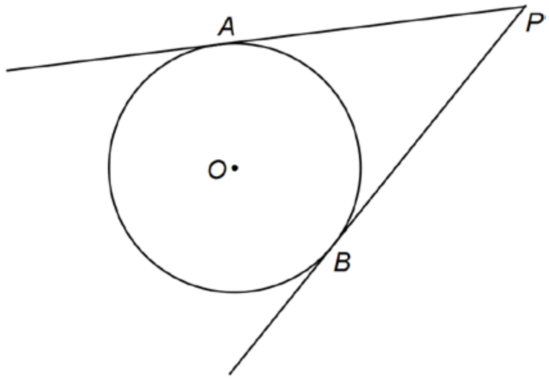
Question 1 **(13 marks)**

(a) (4)



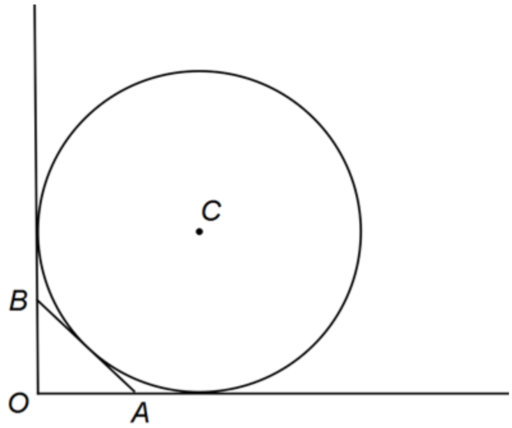
Given PT is tangential to circle centre O at A, use proof by contradiction to show that OA is perpendicular to PQ.

(b) (4)



Given PA and PB are tangents to the circle centre O at A and B respectively, prove that PA and PB are equal in length.
(Hint: draw AO, BO and PO)

(c) (5)

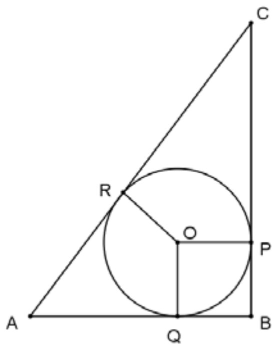


A circle touches the lines OA extended and OB extended and AB where OA and OB are perpendicular.

Determine the relationship between the diameter of the circle centre C and the perimeter of triangle AOB.

Question 2 **(13 marks)**

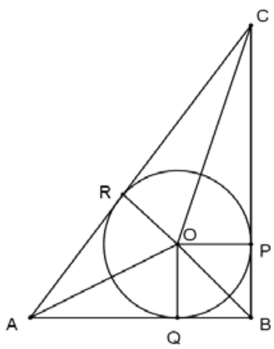
The Tangent – Radius Theorem states that the angle between a tangent and the radius is a right angle.
The Length of Tangents Theorem states that tangents drawn to a circle from an external point are equal in length.



In the diagram above, the circle centre O is inscribed inside the right triangle ABC, i.e. the sides of the triangle are tangential to the circle.

(a) If AB = 3 cm, BC = 4 cm and AC = 5 cm, then determine the radius of the circle. (2)

(b) If AB = 5 cm, BC = 12 cm and AC = 13 cm, then determine the radius of the circle. (1)



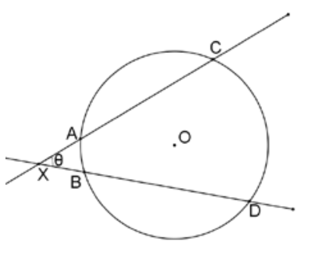
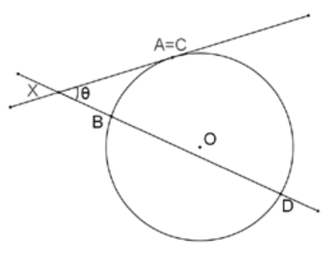
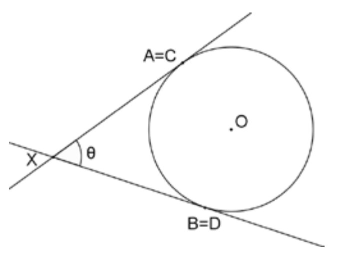
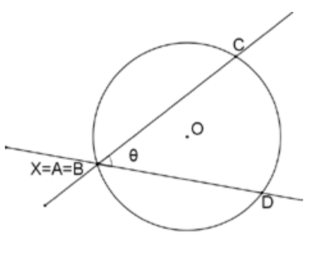
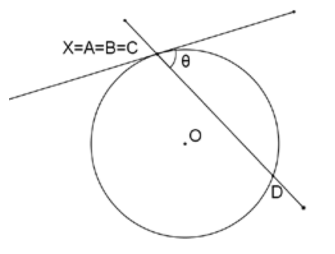
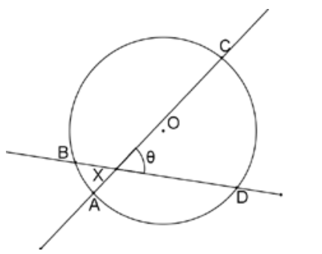
(c) If AB = *a*, BC = *b*, AC = *c* and the radius of the inscribed circle is *r*, by considering the area of ΔABC and equating areas, show that $r = \dfrac{ab}{a+b+c}$. (2)

(d) If $a = 2k(k+1), b = 2k+1$ and $c = 2k^2+2k+1,\ k \in \mathbb{N}$, then show that *a*, *b* and *c* are a Pythagorean triple, i.e. $a^2+b^2=c^2$. (3)

(e) Hence, express *r* in terms of k. (3)

(f) Determine the lengths of the sides of the right triangle whose inscribed circle has a radius of 3 cm. (2)

Question 3 **(15 marks)**

	Secant and Secant	Secant and Tangent	Tangent and Tangent
Out-side			
On			
In-side			

The table above shows six scenarios involving secants and/or tangents intersecting each other and θ is the size of the angle formed by the intersecting lines.

(a) Explain why three of the cells of the table are empty. (3)

Let $\angle COD = \alpha$ and $\angle BOA = \beta$.

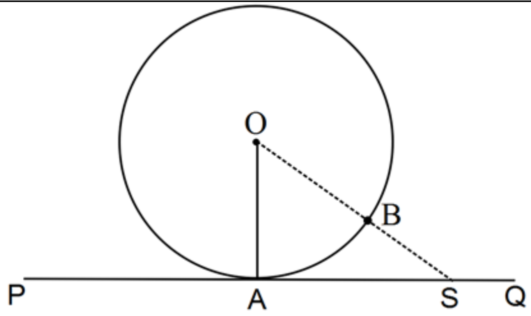
(b) For the case where θ is the angle formed when the secant and the tangent intersect on the circle, express θ in terms of α and/or β. Justify your answer. (2)

(c) For the case where θ is the angle formed when the two secants intersect inside the circle, express θ in terms of α and/or β. Justify your answer. (Hint: draw chord AD) (5)

(d) For the case where θ is the angle formed when the two secants intersect outside the circle, express θ in terms of α and/or β. Justify your answer. (5)

Investigative questions in Topic 1.3
Solutions and marking key

Question 1 (a)



Assume OA is not perpendicular to PQ.
Suppose OS is perpendicular to PQ.

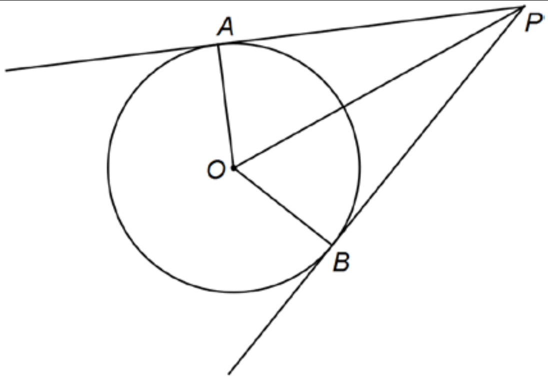
$$\text{In } \Delta\text{OAS}, \ \angle\text{OSA} = 90^\circ \Rightarrow \angle\text{OAS is acute}$$
$$\Rightarrow \text{OA} > \text{OS}$$

But OB = OA and so OB > OS, which is impossible.
So OS cannot be perpendicular to PQ.

Likewise, it can be shown that no other straight line except OA can be perpendicular to PQ.
Therefore OA is perpendicular to PQ.

Marking key/mathematical behaviours	Marks
• Assumes OA is not perpendicular to PQ	1
• Establishes OA > OS	1
• Concludes this gives a contradiction regarding the lengths of OB and OS	1
• Concludes OA is perpendicular to PQ	1

Question 1 (b)



Given: PA and PB are tangents to the circle centre O at A and B respectively

To Prove: PA = PB

Extension to the diagram: Draw AO, BO and PO

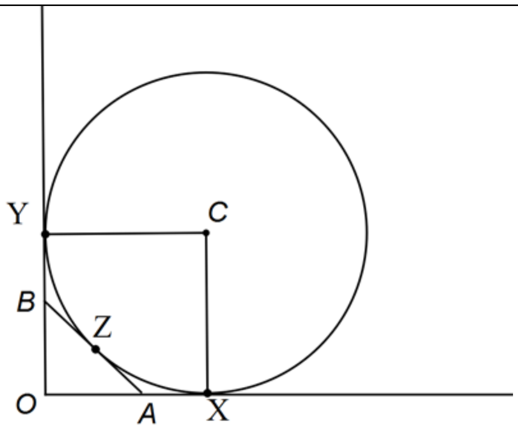
Proof:

In ΔAOP and ΔBOP,

OA = OB	radii
OP = OP	same line segment
$\angle OAP = \angle OBP = 90^\circ$	established in (a)
$\Rightarrow \quad \Delta AOP \cong \Delta BOP$	RHS
$\Rightarrow \quad PA = PB$	corresponding sides congruent Δ's

Marking key/mathematical behaviours	Marks
• Establishes two sides are congruent	1
• Applies result from (a) to establish corresponding right angles	1
• States, with reason, triangles are congruent	1
• Concludes PA = PB	1

Question 1 (c)



Let the circle touch the horizontal and vertical lines at X and Y respectively and the tangent AB at Z.

OX = OY — length of tangents are equal established in (b)

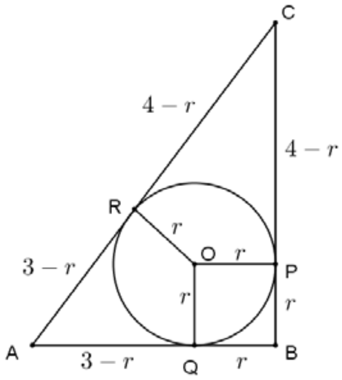
Similarly, AZ = AX and BZ = BY

Diameter of circle = 2 x CX
= 2 x OY
= OX + OY

Perimeter of ΔOAB= OA + OB + AB
= OA + OB + AZ + BZ
= OA + OB + AX + BY
= OA + AX + OB + BY
= OX + OY
= Diameter of circle

Marking key/mathematical behaviours	Marks
• Applies result from (b) to establish OX = OY	1
• Establishes AZ = AX and BZ = BY	1
• Determines diameter of circle	1
• States perimeter of triangle in terms of the three sides	1
• Uses lengths of tangents to establish perimeter of triangle equals diameter of circle	1

Question 2 (a)

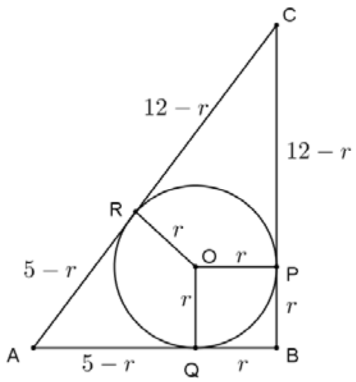


Let radius of circle be r .
Since AC = 5 cm, then

$$3 - r + 4 - r = 5$$
$$\Rightarrow \qquad r = 1 \text{ cm}$$

Marking key/mathematical behaviours	Marks
• Applies length of tangents theorem • Calculates radius	1 1

Question 2 (b)

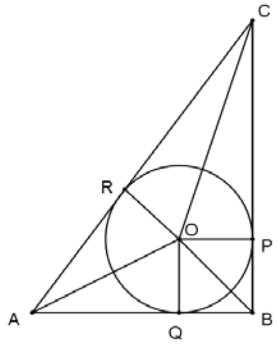


Let radius of circle be r .
Since AC = 13 cm, then

$$5 - r + 12 - r = 13$$
$$\Rightarrow \qquad r = 2 \text{ cm}$$

Marking key/mathematical behaviours	Marks
• Calculates radius	1

Question 2 (c)



Area ΔABC = Area ΔAOB + Area ΔBOC + Area ΔAOC

$$\frac{1}{2}ab = \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr$$

$$ab = r\left(a+b+c\right)$$

$$r = \frac{ab}{a+b+c}$$

Marking key/mathematical behaviours	Marks
• Equates areas • Derives expression for *r*	1 1

Question 2 (d)

Given $a = 2k\left(k+1\right), b = 2k+1$ and $c = 2k^2+2k+1,\ k \in \square$,

$$a^2+b^2 = \left(2k\left(k+1\right)\right)^2 + \left(2k+1\right)^2$$
$$= 4k^2\left(k^2+2k+1\right) + 4k^2+4k+1$$
$$= 4k^4+8k^3+8k^2+4k+1$$

$$c^2 = \left(2k^2+2k+1\right)^2$$
$$= 4k^4+4k^3+2k^2+4k^3+4k^2+2k+2k^2+2k+1$$
$$= 4k^4+8k^3+8k^2+4k+1$$

Hence, $\quad a^2+b^2 = c^2$

Marking key/mathematical behaviours	Marks
• Expands a^2+b^2 • Expands c^2 • Establishes $a^2+b^2=c^2$	1 1 1

Question 2 (e)

Using $r = \dfrac{ab}{a+b+c}$ and $a = 2k(k+1), b = 2k+1 \text{ and } c = 2k^2 + 2k + 1,\ \ k \in \square$,

$$r = \frac{2k(k+1)(2k+1)}{2k(k+1)+(2k+1)+2k^2+2k+1}$$

$$= \frac{2k(2k^2+3k+1)}{2k^2+2k+2k+1+2k^2+2k+1}$$

$$= \frac{4k^3+6k^2+2k}{4k^2+6k+2}$$

$$= \frac{k(4k^2+6k+2)}{4k^2+6k+2}$$

$$= k$$

Marking key/mathematical behaviours	Marks
• Substitutes for a, b and c in terms of k	1
• Simplifies	1
• Expresses *r* in terms of *k*	1

Question 2 (f)

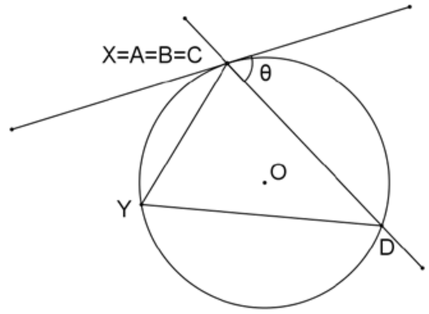
If $r = 3$, then $k = 3$ and $a = 24 \text{ cm}, b = 7 \text{ cm and } c = 25 \text{ cm}$.

Marking key/mathematical behaviours	Marks
• Substitutes $k = 3$	1
• Calculates the lengths of the sides of the triangle	1

Question 3 (a)

A circle divides the plane into three parts – the inside of the circle, the circle itself and the outside of the circle. Secants pass through all three parts whereas tangents do not pass through the circle. Hence, a tangent cannot intersect a secant anywhere inside the circle and tangents can only intersect outside the circle because distinct tangents cannot intersect the circle at the same point.	
Marking key/mathematical behaviours	Marks
• Observes that a circle divides the plane into three parts, a secant passes through all three parts whereas tangents do not pass through the circle. • Concludes that a secant and tangent cannot intersect within the circle. • Concludes that two tangents can only intersect outside the circle.	1 1 1

Question 3 (b)



Draw chords XY and DY.

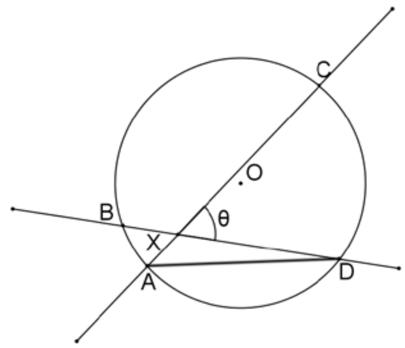
$\angle XYD = \theta$ — Angle in the alternate segment

$\angle XYD = \frac{1}{2}\angle COD$ — Central Angle Theorem

$\Rightarrow \quad \theta = \frac{\alpha}{2}$

Marking key/mathematical behaviours	Marks
• Expresses θ in terms of α • Gives reasons	1 1

Question 3 (c)

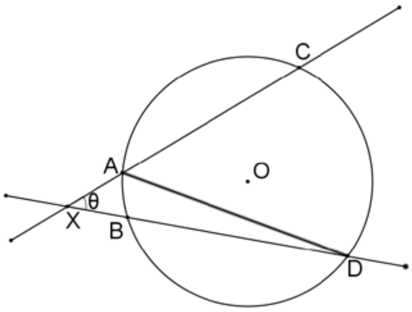


Draw chord AD.

	$\angle CXD = \angle XAD + \angle XDA$	Exterior angle of ΔAXD
	$\angle XAD = \angle CAD$	Same angle
	$\angle CAD = \frac{1}{2}\angle COD$	Central Angle Theorem
$\Rightarrow$	$\angle XAD = \frac{\alpha}{2}$	
	$\angle XDA = \angle BDA$	Same angle
	$\angle BDA = \frac{1}{2}\angle BOA$	Central Angle Theorem
$\Rightarrow$	$\angle XDA = \frac{\beta}{2}$	
$\Rightarrow$	$\angle CXD = \frac{\alpha}{2} + \frac{\beta}{2}$	
$\Rightarrow$	$\theta = \frac{1}{2}(\alpha + \beta)$	

Marking key/mathematical behaviours	Marks
• Applies Exterior Angle Theorem to ΔAXD	1
• Establishes $\angle XAD = \frac{\alpha}{2}$	1
• Establishes $\angle XDA = \frac{\beta}{2}$	1
• Expresses θ in terms of α and β	1
• Gives reasons	1

Question 3 (d)



Draw chord AD.

$\angle CAD = \angle AXD + \angle ADX$ Exterior Angle of ΔAXD

$\Rightarrow \angle AXD = \angle CAD - \angle ADX$

$\angle CAD = \frac{1}{2}\angle COD$ Central Angle Theorem

$\Rightarrow \angle CAD = \frac{\alpha}{2}$

$\angle ADX = \angle ADB$ same angle

$\angle ADB = \frac{1}{2}\angle AOB$ Central Angle Theorem

$\Rightarrow \angle ADB = \frac{\beta}{2}$

$\Rightarrow \angle ADX = \frac{\beta}{2}$

$\Rightarrow \angle AXD = \frac{\alpha}{2} - \frac{\beta}{2}$

$\Rightarrow \theta = \frac{1}{2}(\alpha - \beta)$

Marking key/mathematical behaviours	Marks
• Applies Exterior Angle Theorem to ΔAXD	1
• Establishes $\angle CAD = \frac{\alpha}{2}$	1
• Establishes $\angle ADX = \frac{\beta}{2}$	1
• Expresses θ in terms of α and β	1
• Gives reasons	1